VIOLATION OF RICHARDSON'S CRITERION VIA INTRODUCTION OF A MAGNETIC FIELD

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ABSTRACT

Shear flow instabilities can profoundly affect the diffusion of momentum in jets, stars, and disks. The Richardson criterion gives a sufficient condition for instability of a shear flow in a stratified medium. The velocity gradient V' can only destabilize a stably stratified medium with squared Brunt–Väisälä frequency N^2 if $V'^2/4 > N^2$. We find this is no longer true when the medium is a magnetized plasma. We investigate the effect of stable stratification on the magnetic field and velocity profiles unstable to magneto-shear instabilities, i.e., instabilities which require the presence of both magnetic field and shear flow. We show that a family of profiles originally studied by Tatsuno & Dorland remains unstable even when $V'^2/4 < N^2$, violating the Richardson criterion. However, not all magnetic fields can result in a violation of the Richardson criterion. We consider a class of flows originally considered by Kent, which are destabilized by a constant magnetic field, and show that they become stable when $V'^2/4 < N^2$, as predicted by the Richardson criterion. This suggests that magnetic free energy is required to violate the Richardson criterion. This work implies that the Richardson criterion cannot be used when evaluating the ideal stability of a sheared, stably stratified, and magnetized plasma. We briefly discuss the implications for astrophysical systems.

Key words: hydrodynamics – instabilities – magnetohydrodynamics (MHD) – stars: magnetic field – stars: rotation

1. INTRODUCTION

Rotation plays an important role in the structure and evolution of stars. Although rotation directly modifies hydrostatic equilibrium only in the most rapid rotators, it drives large-scale circulation, modifies the structure of convection and the nature of convective transport, and is a key component of magnetic dynamos. These phenomena in turn modify the rotation through a complex interplay of nonlinear processes.

Shear flow instability is one of the mechanisms through which rotation influences and is influenced by its environment. The motion associated with the instability generates stresses, which react back on the flow and drive it toward a stable state. If the amplitude of the unstable perturbations is sufficiently large, the motions become turbulent. Shear flow instability and shear flow turbulence can amplify magnetic fields and mix chemical species, in addition to modifying the rotation profile itself.

In the case of the Sun, and possibly other low-mass main-sequence stars, the most likely venue for shear flow instability is the so-called tachocline, the region of strong shear just below the base of the convection zone (see Gough 2007, for a review). Although the mechanisms which maintain the tachocline are still uncertain, it is almost certainly a component of the solar dynamo, and its existence has implications for the way the convection zone, which is spun down by the solar wind, is coupled to the radiative core. The tachocline may be subject to purely hydrodynamic instabilities (Rashid et al. 2008; Kitchatinov & Rüdiger 2009), global magnetohydrodynamics (MHD) instabilities driven by the latitudinal structure of the field (Gilman & Fox 1997; Gilman et al. 2007) magnetorotational instabilities (Ogilvie 2007), and, if hydromagnetic forces are large enough, magnetic buoyancy instabilities (Silvers et al.

2009; Vasil & Brummell 2009). All these instabilities could modify the tachocline's structure.

Massive stars, which evolve quickly and tend to rotate rapidly, are potentially more profoundly affected by shear flow instability. The past two decades have witnessed significant advances in understanding how the internal rotation of massive luminous stars shapes, and is shaped by, their evolution (see Maeder & Meynet 2000, and references therein for a comprehensive review). Rapidly rotating massive stars follow bluer, more luminous evolutionary tracks in the Hertzsprung–Russell diagram (HRD) than non-rotating equivalents, because strong meridional circulation injects fresh hydrogen fuel into the convective core (see, e.g., Meynet & Maeder 2000). This rotational mixing brings CNO-cycle nucleosynthetic products from the stars' cores to their surfaces, leading to changes in photospheric abundance ratios (e.g., Talon et al. 1997).

The prevailing view of rotation in massive stars is based on a canonical narrative developed by Zahn (1992). In this scenario, turbulent diffusion of angular momentum is highly anisotropic, with much stronger transport in the horizontal direction than the radial one. This leads to a "shellular" rotation profile, in which the angular velocity is constant on spherical shells. The exchange of angular momentum between these shells is then mediated by a combination of meridional circulation, convection (in convective zones), and radial turbulent diffusion. The turbulence itself is driven by secular shear instability (Maeder & Meynet 2000), which grows on a thermal timescale (see also Maeder 1995; Maeder & Meynet 1996; Talon & Zahn 1997).

Recent studies have considered the role that magnetic fields might play in modifying angular momentum transport (e.g., Maeder & Meynet 2004). Generally, these studies of the impact of magnetic fields have focused around contributions to the radial angular momentum diffusivity arising from the field stiffness (Petrovic et al. 2005). However, as Spruit (1999) has discussed, a field can also introduce new instabilities that play a

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role in angular momentum transport. In this paper, we explore a hitherto-overlooked magnetic-mediated instability, whereby the presence of a *horizontal* field can destabilize a stratified shear layer that—according to the Richardson criterion—would otherwise be stable.

The paper is organized as follows. First we will briefly discuss shear flow instabilities in Section 2. In Section 3, we set up the eigenvalue problem which determines the linear stability of an MHD shear flow in a stratified medium. We review previous analytic results in Section 4, and describe our numerical methods for solving the eigenvalue problem in Section 5. Starting in Section 6 we examine specific examples, first adding stratification to the linear velocity and parabolic magnetic field example considered in a recent paper by Tatsuno & Dorland (2006, hereafter TD06). Our key result is that sufficiently strong parabolic magnetic fields can yield instability for arbitrarily strong stratification, in violation of the Richardson criterion. We consider and extend a family of velocity profiles which Kent (1968, hereafter K68) showed can be destabilized by a constant magnetic field in Section 7. In contrast to the parabolic magnetic field case, it seems that the introduction of a constant magnetic field cannot result in a violation of the Richardson criterion. This suggests that the free energy of an inhomogeneous magnetic field is essential to breaking the Richardson criterion. We discuss possible applications to rotating stars in Section 8 and conclude in Section 9.

2. INTRODUCTION TO SHEAR FLOW INSTABILITIES

The best known shear flow instability is the hydrodynamic Kelvin–Helmholtz instability. The Kelvin–Helmholtz instability has been studied extensively. Perhaps the most famous result is the inflexion point criterion, stating that a necessary condition for instability is the presence of an inflexion point in the velocity profile (see, for example, Drazin & Reid 1981). Others have also given necessary conditions for instability, making extra assumptions on the flow profile (Lin 1955; Howard 1961; Rosenbluth & Simon 1964).

Many have worked to extend parts of these results to MHD shear instabilities. It is well known that a sufficiently strong magnetic field stabilizes the Kelvin–Helmholtz instability (Chandrasekhar 1961). It was shown years ago, but is perhaps less well known, that a magnetic field can destabilize an otherwise stable shear flow (K68). In particular, an inflexion point is no longer necessary for shear instability. In hydrodynamics vorticity is frozen into the flow, ensuring that perturbations are stable when there is no inflexion point (Lin 1955), but the presence of a magnetic field can break the vorticity frozen-in condition, relaxing the inflexion point criterion. K68 constructed a family of flow profiles which are marginally stable in the absence of a magnetic field and destabilized by a uniform field parallel to the direction of flow. TD06 studied how a linear flow profile, which has no inflexion point and is marginally stable, can be destabilized by a particular family of magnetic field profiles. In particular, TD06 find that a parabolic magnetic field can render a linear velocity profile unstable.

In this paper, we add a new piece of physics to the analysis: density stratification. We employ the Boussinesq approximation and assume that the plasma is stably stratified, i.e., the squared Brunt–Väisälä frequency, N^2, is positive. In hydrodynamics, the Richardson criterion provides a sufficient condition for the stability of a shear flow in a stratified medium (see, for example, Drazin & Reid 1981). The interchange of two fluid elements at different heights can release kinetic energy from the flow.

A necessary condition for instability is that the gravitational energy required for the interchange must be less than the kinetic energy released. However, in the presence of an inhomogeneous magnetic field, energy can also be extracted from the magnetic field, even if the field would be stable in the absence of shear flow. Our main result is that the Richardson criterion no longer holds for inhomogeneous magnetic fields.

We will only consider the effect of stable stratification on magneto-shear instabilities. However, Tatsuno et al. (2003) studied how a shear flow can destabilize a homogeneous magnetic field in the presence of an *unstable* density gradient. They found that a linear (Couette) velocity profile can be destabilizing when the velocity shear was not too strong. Their result is similar to ours in the sense that the system is maximally destabilized when the velocity gradient, magnetic field, and density stratification all have comparable strength.

In this paper, we consider only ideal instabilities, i.e., we set the resistive, viscous, and thermal diffusivities to zero. Diffusive effects could unleash a host of additional instabilities such as tearing modes (e.g., Furth et al. 1963), doubly diffusive modes (e.g., Schmitt & Rosner 1983), and secular shear instabilities (e.g., Maeder & Meynet 2000). Although such instabilities are important in their own right, in this paper we focus entirely on dynamical instabilities.

3. BASIC EQUATIONS

The time evolution of an ideal, incompressible plasma is given by

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla} \mathbf{V} \right) = -\boldsymbol{\nabla} \left(p + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B} - g\rho \mathbf{e}_z,$$
(1)

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}),$$
(2)

$$0 = \boldsymbol{\nabla} \cdot \mathbf{V},$$
(3)

$$0 = \boldsymbol{\nabla} \cdot \mathbf{B},$$
(4)

$$0 = \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla} \rho,$$
(5)

where the symbols have their usual meanings. Equation (1) is the momentum equation, Equation (2) is the induction equation, Equation (3) enforces incompressibility, Equation (4) is the divergenceless magnetic field condition, and Equation (5) is the continuity equation. We will write the unit vectors in the x, y, and z directions as $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, respectively. The gravitational strength is parameterized by g, and gravity is assumed to point in the $-\mathbf{e}_z$ direction. We denote background velocity and magnetic fields with capital letters, and then perturb the background fields with fields denoted with lower case letters, except that the background density is denoted by ρ, and the perturbed density by $\hat{\rho}$. We assume that the background quantities ρ, $\mathbf{V}$, and $\mathbf{B}$ all are the functions of only z, and that our domain is the volume between $z = -z_0$ and $z = +z_0$ with "free-slip," perfectly conducting boundary conditions in the z direction, and periodic boundary conditions in the x and y directions. By "free-slip," we mean no constraint on perturbed quantities in the x and y directions at the walls, but that perturbations have no z component at the walls. These are the boundary conditions adopted by TD06 (who termed them "no-slip" which is not correct—as will be shown in Section 6.3, the perturbations slip

along, but do not penetrate, the walls). Next, we assume that **V** is oriented toward only one direction throughout the domain, which we define to be the x direction. Thus, we take

$$\mathbf{V} = (V(z), 0, 0), \tag{6}$$

in Cartesian coordinates. The background magnetic field **B** is

$$\mathbf{B} = (B_x(z), B_y(z), 0) \tag{7}$$

in Cartesian coordinates. The background fields are assumed to be in equilibrium, so we have that

$$\boldsymbol{\nabla} \left(p + \frac{B^2}{2\mu_0} \right) + g\rho \mathbf{e}_z = 0. \tag{8}$$

Equation (8) specifies an integral equation for the background pressure p for arbitrary B and ρ. The induction and continuity equations for the background fields are automatically satisfied by the geometry we have imposed.

Now assume the perturbation fields all have the form

$$f(x, y, z, t) = f(z) \exp(ik_x x + ik_y y - ik_x ct). \tag{9}$$

We will take $\mathbf{k} \equiv k_x \mathbf{e}_x + k_y \mathbf{e}_y$, $k = |\mathbf{k}|$ and $\hat{\mathbf{k}} = \mathbf{k}/k$. In many applications, the density gradient ρ' is small in comparison to the velocity gradient V'—where prime denotes differentiation with respect to z—but the strength of gravity g is large. Assuming this, we recover the Boussinesq approximation, in which we drop terms proportional to ρ' alone, but keep terms proportional to $g\rho'$. These assumptions yield the following eigenvalue problem for ξ, the plasma displacement in the z direction:

$$\left(\left[k_x^2 (V - c)^2 - k^2 A^2 \right] \xi' \right)'$$
$$- k^2 \left[k_x^2 (V - c)^2 - k^2 A^2 \right] \xi + k^2 N^2 \xi = 0, \tag{10}$$

where $A \equiv \hat{\mathbf{k}} \cdot \mathbf{B}/\sqrt{\rho\mu_0}$ is the Alfvén velocity, and $N^2 \equiv g\rho'/\rho$ is the Brunt–Väisälä frequency in the Boussinesq approximation. To simplify our analysis, we assume that N^2 is constant throughout the domain, which corresponds to the exponentially decaying density profile. When computing the Alfvén velocity, the Boussinesq approximation will allow us to consider ρ to be a constant. The boundary conditions are that $\xi = 0$ at the boundaries at $z = -z_0$ and $z = +z_0$.

There is an asymmetry in how velocity shear, magnetic fields, and density stratification depend on the wavenumber **k**. For $\mathbf{k} = k_y \mathbf{e}_y$, $k_x = 0$ and the velocity shear is irrelevant (note that $k_x c$, the growth rate, could still be finite). The purpose of this paper is to examine the interplay between velocity and magnetic fields, so we will not consider this case. Also note that the Alfvén velocity, as it occurs in Equation (10), is a function of $\hat{\mathbf{k}}$. For example, if **B** is constant in the z direction, there exists a $\hat{\mathbf{k}}$ for which $A = 0$, so the magnetic field would have no effect on such a perturbation. The strength of gravity in relation to shear flow contains a factor of k^2/k_x^2. Thus, gravity is maximally destabilized by shear flows when $k_y = 0$.

Consider an eigenvalue problem for the magnetic field **B**, velocity V, Brunt–Väisälä frequency N^2, and wavenumber $\mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y$, with $k_y \neq 0$. We will show that this eigenvalue problem is equivalent to another eigenvalue problem with $k_y = 0$, but with different **B**, N^2, and k_x. Define $\mathbf{B}' \equiv \mathbf{e}_x \mathbf{k} \cdot \mathbf{B}/k_x$, $N'^2 \equiv k^2 N^2/k_x^2$, and $\mathbf{k}' \equiv k\mathbf{e}_x$. Then the magnetic field $\mathbf{B}'$,

velocity V, Brunt–Väisälä frequency N'^2, and wavenumber $\mathbf{k}'$ have the same eigenvalue equation as above. Thus, finite k_y is equivalent to $k_y = 0$, if one appropriately rotates and augments the magnetic field, and increases the density stratification. With this in mind, we will consider the $k_y = 0$ case in the remainder of this paper, for which the eigenvalue equation reduces to

$$\left(\left[(V - c)^2 - A^2 \right] \xi' \right)' - k^2 \left[(V - c)^2 - A^2 \right] \xi + N^2 \xi = 0. \tag{11}$$

The eigenvalue Equation (11) possesses some symmetries. First, the sign of A is unimportant, so changing the sign of the magnetic field does not change the problem. Another symmetry is translational: taking $V \rightarrow V + \Delta V$ and $c \rightarrow c - \Delta V$ corresponds to Galilean transformations. Thus, without loss of generality, we can and do put ourselves in a frame in which $V(0) = 0$. To make the problem more tractable, we add additional symmetries to the equation by postulating that A is even in z and V is odd. There is a rescaling symmetry: Equation (11) remains invariant under

$$\begin{aligned}
z &\rightarrow z/z_0, \\
V &\rightarrow V/z_0, \\
A &\rightarrow A/z_0, \\
k &\rightarrow kz_0, \\
c &\rightarrow c/z_0.
\end{aligned} \tag{12}$$

Note that N^2 and Ri are left unchanged under this transformation.

There is also structure in the eigenvalues. In general, c and ξ are complex: $c = c_r + ic_i$, $\xi = \xi_r + i\xi_i$. We ignore the singular $c_i = 0$ case. If ξ_c is an eigenfunction with eigenvalue c, then ξ_c^*, the complex conjugate of ξ_c, is a solution to Equation (11) with eigenvalue c^*. Thus, eigenvalues come in complex conjugate pairs, regardless of the symmetry properties of A and V. Assuming that A is even and V is odd, we can show that if c is an eigenvalue, then $-c$ is also an eigenvalue, with eigenfunction $\xi_c(-z)$.

Numerically, we only find eigenvalues with $c_r = 0$ and with the following eigenfunction symmetry. If we normalize the eigenfunction ξ such that $\xi(0) = 1$, then ξ_r is even and ξ_i is odd. In Sections 6 and 7, we assume that $c_r = 0$, and the eigenfunction has this symmetry. These properties are linked. If we multiply Equation (11) by ξ^* and integrate over the domain the result is

$$\int_{-z_0}^{z_0} \left[(V - c)^2 - A^2 \right] \left(\left| \frac{d\xi}{dz} \right|^2 + k^2 |\xi|^2 \right) - N^2 |\xi|^2 dz = 0. \tag{13}$$

The imaginary part of Equation (13) is

$$2ic_i \int_{-z_0}^{z_0} (c_r - V) \left(\left| \frac{d\xi}{dz} \right|^2 + k^2 |\xi|^2 \right) dz = 0. \tag{14}$$

If the real and imaginary parts of ξ each have definite parity, the term proportional to V in Equation (14) vanishes. Therefore, $c_r c_i \equiv 0$, and unstable modes have $c_r = 0$. This result is useful in searching for unstable modes, as described in Section 5.

We find that generally the growth rate $c = ic_i$ is small in comparison to V, which is $\mathcal{O}(1)$. When $V^2 = A^2$, the coefficient of the ξ'' term in Equation (11) goes to $|-c_i^2| \ll 1$. Thus, the equation becomes "almost singular" when $|V| = |A|$, and becomes actually singular when $c = 0$. The "almost singularities"

are characterized by large gradients in the eigenfunctions, as is shown in Sections 6 and 7.

We will often consider the limit $k^2 = 0$. When $k^2 = 0$, the growth rate, kc, is formally zero. However, one can view the eigenvalue c as a function of the various parameters A, V, k^2, and N^2. We assume that $c(k^2)$ is analytic about $k^2 = 0$, so our results for the $k^2 = 0$ case still hold in a neighborhood of $k^2 = 0$. Thus, when we consider $k^2 = 0$, we are really taking the limit as k becomes small. The k^2 term in Equation (11) is only important when it is comparable to the scale heights of the velocity and magnetic fields and the perturbation ξ. Numerically, we find that $kz_0 < 0.1$ is "small" for the examples presented in this paper.

4. REVIEW OF ANALYTIC RESULTS

Shear flow instabilities are global instabilities. Thus, the two categories of analytic results—necessary conditions for instability and sufficient conditions for instability—can be viewed as local and global conditions. Necessary conditions for instability give criteria which must be satisfied in at least one spot in the domain, whereas the sufficient conditions for instability are global criteria involving integrals over the domain. We present a short overview of the analytic results regarding the linear stability of shear flows. We begin by discussing shear flows alone, and then add stratification, a magnetic field, and then both. The zero magnetic field and zero density gradient cases can be viewed as limits of the more general problem.

4.1. Shear Flow Instabilities

Probably the best known result is the inflexion point criterion, which states that V'' must have a zero in the domain for there to be instability. This is a local, necessary condition. There are several physical interpretations of the inflexion point criterion. Consider the Reynolds stress of the perturbation, $\tau = -\rho \overline{v_x v_z}$, where the bar denotes averaging with respect to x. Assuming $c \neq 0$, one can show that $d\tau/dz$ has a zero iff V'' has a zero (for instance, in Lin 1955, or K68). Since $\tau = 0$ at the boundaries, when $c \neq 0$, we must have that V'' has a zero. Lin (1955) has proposed an alternate interpretation considering vorticity. A zero in V'' corresponds to an extremum in vorticity, and Lin has shown that perturbations feel a restoring force unless they are at an extremum of vorticity.

The inflexion point theorem is useful because it rules out a large class of velocity profiles as stable. However, it cannot be used to show that a particular shear flow is unstable. Rosenbluth & Simon (1964) were able to prove a necessary and sufficient condition for instability by using the additional assumptions that V'' has a single zero and V is monotonic. Under these assumptions, V is unstable in $z_1 \leqslant z \leqslant z_2$ if and only if

$$\left. \frac{1}{V'(V_c - V)} \right|_{z_1}^{z_2} - \int_{z_1}^{z_2} \frac{V''}{V'^3(V - V_c)} dz > 0, \qquad (15)$$

where V_c is the velocity at the inflexion point. This result is derived for the $k^2 = 0$ case. A priori, it seems that there could be velocity profiles which are unstable for $k^2 > 0$ but stable for $k^2 = 0$. Then an instability condition for $k^2 = 0$ would be only sufficient for instability. This is addressed by a theorem of Lin (1955) which shows that under the assumptions of Rosenbluth & Simon, velocity profiles which are unstable for $k^2 > 0$ are also unstable for $k^2 = 0$.

4.2. Shear Flow Instabilities in a Stratified Medium

The key stability result for stratified media is the Richardson criterion, a necessary condition for the instability of a shear flow in a stratified medium. If

$$\mathrm{Ri} \equiv \frac{N^2}{V'^2} > \frac{1}{4} \qquad (16)$$

everywhere, then there is stability. A physical interpretation (see, for example, Chandrasekhar 1961 or Drazin & Reid 1981) is that if exchanging fluid elements at slightly different heights increases the potential energy more than it decreases the kinetic energy, then the perturbation is stable.

Provided that $\mathrm{Ri} < 1/4$, we have that

$$k^2 c_i^2 \leqslant \max \left(\frac{1}{4} V'^2 - N^2 \right). \qquad (17)$$

This result by Howard (1961) follows from the proof of the Richardson criterion and is also discussed in Drazin & Reid (1981).

4.3. Magneto-shear Instabilities

Magnetic fields can both stabilize and destabilize shear flows. First, we consider their stabilizing effect. Perturbations which bend magnetic field lines induce a restoring magnetic tension force. A classic result is that in a constant density medium, the vortex sheet $V(z) = -U$ for $z < 0$ and $V(z) = +U$ for $z > 0$ for some constant U, is stabilized by a magnetic field A if and only if $A^2 > V^2$ (Chandrasekhar 1961). This step function velocity profile is the limiting distribution of $V(z) = U_0 \tanh(z/a)$ as $a \to 0$. Keppens et al. (1999) have investigated the hyperbolic tangent V case with a constant magnetic field, including compressibility, and found the magnetic field stabilizing. These results were qualitatively similar to those by Chandrasekhar, which is expected because a constant magnetic field has no length scale (or it has an infinite length scale), so it cannot tell the difference between the $a \to 0$ and a finite case.

Keppens et al. also found that the addition of a non-uniform magnetic field could be destabilizing. When they added a small field $A(z) = -A_0$ for $z < 0$ and $A(z) = A_0$ for $z > 0$, they found that the growth rate increased, and was even larger when A reversed smoothly. Although their calculation, unlike ours, includes compressibility, there is one robust effect which is always present: magnetic fields allow transfer of vorticity between fluid elements. The loss of the frozen-in vorticity constraint changes the range of motions allowed in the plasma, and yielding instability.

We now review some general results on magneto-shear instabilities in order to understand how the Richardson criterion can be violated by the introduction of a magnetic field.

The necessary and sufficient instability condition of Rosenbluth & Simon (1964; Equation (15)) has been generalized to the MHD case by K68 and Chen & Morrison (1991). Both arguments use that when $k^2 = 0$, there is an exact solution to Equation (11),

$$\xi(z) = \int_{z_1}^{z} \frac{dz'}{(V - c)^2 - A^2}, \qquad (18)$$

and then define

$$f(c) \equiv \int_{z_1}^{z_2} \frac{dz}{(V - c)^2 - A^2} = \xi(z_2). \qquad (19)$$

The eigenvalues of Equation (11) are then just the zeros of $f(c)$, and one can search for instabilities by implementing Nyquist's method to determine if there are any zeros of $f(c)$ for $c_i > 0$. Nyquist's method is an application of the argument principle (see, for instance, Gamelin 2001), which states that the integral of the argument of $f(c)$ on the boundary ∂D of some region D is equal to $2\pi(N_0 - N_\infty)$, where N_0 is the number of zeros of $f(c)$ in D, and N_∞ is the number of poles of $f(c)$ in D. In our case, we assume $N_\infty = 0$, so counting the number of times $f(c)$ wraps around the origin tells us how many zeros, i.e., unstable modes, there are. Further discussion of Nyquist's method can be found in Krall & Trivelpiece (1973).

Nyquist's method can only be applied if we know what contour to use. The real part of c can be bounded by extending an important hydrodynamic result by Rayleigh. It can be shown (Hughes & Tobias 2001) that c_r must lie in the range of V, so the contour in c space is bounded by $V_{\min} < c_r < V_{\max}$. The lower bound for c_i is 0^+, and the upper bound can be recovered by modifying Howard's semicircle theorem (Howard 1961). In the hydrodynamic case, Howard showed (see, for instance, Drazin & Reid 1981) that

$$\left[c_r - \tfrac{1}{2}(V_{\max} + V_{\min})\right]^2 + c_i^2 \leqslant \left[\tfrac{1}{2}(V_{\max} - V_{\min})\right]^2. \quad (20)$$

Thus, we have that $c_i \leqslant 1/2(V_{\max} - V_{\min})$. Hughes & Tobias (2001) have shown that in MHD, we have the two inequalities

$$(V^2 - A^2)_{\min} \leqslant c_r^2 + c_i^2 \leqslant (V^2 - A^2)_{\max}, \quad (21)$$

and

$$\left[c_r - \tfrac{1}{2}(V_{\max} + V_{\min})\right]^2 + c_i^2 \leqslant \left[\tfrac{1}{2}(V_{\max} - V_{\min})\right]^2 - (A^2)_{\min}. \quad (22)$$

This gives an even stronger upper bound on c_i, that

$$c_i \leqslant \sqrt{(1/2(V_{\max} - V_{\min}))^2 - (A^2)_{\min}}. \quad (23)$$

These two inequalities can be used to show stability, if one can show that there are no c which simultaneously satisfy both inequalities.

Chen & Morrison (1991) used Nyquist's method to provide a sufficient condition for instability for flows in which V is even, and A is either odd or even. They showed that

$$\Re \int_{-z_0}^{z_0} \frac{dz}{(V - i\epsilon)^2 - A^2} > 0 \quad (24)$$

as $\epsilon \to 0$ is sufficient for instability. Note that it is not assumed that V has an inflexion point.

K68 considered the effects of a small, constant magnetic field on a stable velocity profile. He showed that when V'' has a single zero, and there exist points y_s, y_t such that the velocities at these points, V_s, V_t satisfy $V_s - V_t = 2A$ and $V_s' - V_t' = 0$, then

$$M(A) \equiv \wp \int_{z_1}^{z_2} \frac{dz}{(V - c_0)^2 - A^2} > 0, \quad (25)$$

implies instability. Here, c_0 is defined by $c_0 = (V_s + V_t)/2$, and $\wp$ denotes the principal value of the integral. For small A, c_0 is the velocity at the inflexion point, but as A increases, it can deviate somewhat. For a marginally stable velocity profile, we have $M(0) = 0$. In the remainder of this section, we will use $\dot{}$ (dot)

to denote derivative with respect to A. In the limit $A \to 0$, we have $\dot{M}(A) \to 0$. Thus, to evaluate the stability of V to infinitely small A, we need to consider $\ddot{M}(0)$, which Kent shows is given by

$$\ddot{M}(0) = 2\ddot{c}_r(0) \int_{z_1}^{z_2} \frac{dz}{(V - C_0)^3} + 2\int_{z_1}^{z_2} \frac{dz}{(V - C_0)^4}, \quad (26)$$

where

$$\ddot{c}_r(0) = -\frac{C_0^{(4)}}{3C_0 C_0^{(3)}}. \quad (27)$$

This criterion is useful because one can change variables to integrate over V, and if $C_0 = 0$ and $\omega(V) := dz/dV$ is even, then

$$\ddot{M}(0) = \int_{V_1}^{V_2} \frac{\omega \, dV}{V^4}, \quad (28)$$

where $V_i = V(z_i)$. Although these conditions are sufficient for instability, they are not necessary. Unlike in the hydrodynamic case, there can be unstable modes for finite k^2 for a velocity profile which is stable at $k^2 = 0$ (K68).

Another way to tackle the general problem with arbitrary velocity and magnetic field profiles is to attempt to extend the physical arguments behind the inflexion point criterion to the MHD problem. In the MHD problem, one must consider both the Reynolds and Maxwell stresses, so the total stress is given by

$$\tau_{\text{tot}} = -\rho \overline{v_x v_z} + \overline{b_x b_z}. \quad (29)$$

A necessary condition for instability is still $d\tau_{\text{tot}}/dz = 0$ somewhere in the flow. K68 has shown that this condition can be written as

$$\Im\left[|X|\frac{X'}{X}\right]' = 0, \quad (30)$$

or

$$\Im\left[\frac{2XX'' - X'^2}{4X^2}\right] = 0, \quad (31)$$

where $X \equiv (V - c)^2 - A^2$. Unfortunately, these (equivalent) conditions are not as useful as the inflexion point criterion because they depend on both the flow profile and the growth rate. Thus, one needs to check Equations (30) or (31) for all possible c. This condition seems to be fairly weak, and is satisfied by many stable profiles.

4.4. Magneto-shear Instabilities in a Stratified Medium

The addition of a magnetic field to a shear flow in a stratified medium makes the problem significantly more complex. The Richardson criterion is no longer valid, but it can be generalized. We have carried out the same analysis used to derive the Richardson criterion, but included magnetic fields. The result is that if

$$0 > \frac{1}{c_i}\Im\left(\frac{2ZZ'' - Z'^2}{4Z^2} + \frac{V'Z'}{Z} + \frac{\frac{V'^2}{4} - \frac{N^2}{Z}}{(V - c)}\right) \quad (32)$$

everywhere in the domain, then the system is stable. Here, $Z \equiv 1 - A^2/(V - c)^2$. Similar to the generalization of the inflexion point criterion (Equations (30) and (31)), this condition involves c. This condition also seems to be weak.

Although we normally assume that $c_r = 0$, this condition can be relaxed, and we can find bounds for c_r. The argument by

Hughes & Tobias (2001) mentioned in Section 4.3 still holds when stratification is introduced and shows that c_r must lie within the range of V. This bound on c_r is valid with and without magnetic field, and with and without stratification.

5. NUMERICAL METHODS

Because the problem is global, analytic results exist only in cases with particular symmetries (i.e., $k^2 = 0$ or $N^2 = 0$), so we must generally solve for stability numerically. We have implemented three numerical methods for solving the eigenvalue problem, Equation (11). In the first, we discretize the equation onto a Chebyshev grid, and use a finite-dimensional approximation for the differential operator. Then Equation (11) can be rewritten as a generalized finite-dimensional eigenvalue equation:

$$
\gamma \begin{pmatrix} \mathcal{D} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_z \\ b_z \\ \hat{\rho} \end{pmatrix}
$$
$$
= \begin{pmatrix} -ik_x V\mathcal{D} + ik_x V'' & ikA\mathcal{D} - ikA'' & -N^2 k^2 \\ ikA & -ik_x V & 0 \\ 1 & 0 & -ik_x V \end{pmatrix} \begin{pmatrix} v_z \\ b_z \\ \hat{\rho} \end{pmatrix},
$$
(33)

where $\mathcal{D} \equiv \partial_z^2 - k^2$. Matlab was used to solve this finite-dimensional eigenvalue problem. This approach was useful when we did not require high resolution. This method was not able to resolve the large gradients in the eigenfunctions that sometimes appeared when $|V| = |A|$.

Another strategy, for $k = 0$, was implementing Nyquist's method. We used Mathematica to calculate $f(c)$, as defined in Equation (19) for various c. As mentioned in Section 4.4, we know that c_r lies between the minimum and maximum of V. The advantage of Nyquist's method is that we need not assume that c is imaginary. We picked the rectangle with vertices at $i\epsilon + V_{max}$, $i\epsilon + V_{min}$, $ia + V_{min}$, and $ia + V_{max}$ as the contour, with a of order 1 and ϵ small. If one plots $f(c)$, where c traverses this contour, it is easy to see if there are any unstable modes with c in this contour. We varied the size of the rectangular contour to find the exact eigenvalues. For the examples presented below in Sections 6 and 7, eigenvalues were always purely imaginary, and the eigenfunctions had the symmetry properties described in Section 3.

Finally, we used a finite difference relaxation code to integrate across the domain. We assumed that c was imaginary, and integrated Equation (11) over the domain for c between $i\epsilon$ and ia for a of order 1 and ϵ small, in logarithmic steps. When the real part of $f(c)$ changed sign between two consecutive steps, the secant method was used to find the zero in the real part of $f(c)$, which corresponds to a zero in $f(c)$. This algorithm was the most efficient, but makes the assumption that the eigenvalues are purely imaginary. As mentioned in Section 3, we have not found any eigenvalues with non-vanishing real part using the other two methods mentioned above, so this seems to be a valid assumption.

All three numerical methods give similar results in cases where we used more than one.

6. LINEAR V, PARABOLIC A

In this section, we add density stratification to the linear velocity and parabolic magnetic field profiles considered by TD06. The main result is that we find instability even when $V'^2/4 < N^2$ everywhere, i.e., when the Richardson criterion

predicts stability. We believe this is because the magnetic field provides another free energy source for the instability. At $k^2 = 0$, there are magnetic field profiles which are unstable for arbitrarily large N^2, but when $k^2 > 0$, there is only a finite range of N^2 which are unstable for the profiles considered here.

6.1. The Field and Flow Profiles

Consider the following velocity and magnetic field profiles in a domain from $z = -1$ to $z = +1$:

$$V(z) = z,$$
(34)

$$A(z) = (1 - \alpha)z^2 + \alpha.$$
(35)

These are the fields considered in Section III.A.1 of TD06 (where we call their α_1 parameter α). The magnetic field is a parabola with $A(0) = \alpha$ and $A = 1$ at the boundaries.

An important characteristic of these profiles is that neither the magnetic field nor the velocity profile are unstable by themselves. The instability is truly a magneto-shear instability, as both magnetic field and shear flow play a part in rendering the profiles unstable. In this respect, this example is different from those considered by others in which a magnetic instability is stabilized by gravity (Dikpati et al. 2009), a magnetic layer destabilizes a stratified medium (Newcomb 1961), or magnetic field and shear flow modify a buoyancy instability (Howes et al. 2001).

These profiles can be viewed as local approximations to a wide range of field and flow profiles. The parabolic magnetic field profile is valid locally whenever B has an extremum, which we take to be at $z = 0$. As mentioned in Section 3, taking $A \to -A$ does not change the problem, so although we are considering a local minimum, the exact same results hold for $A(z) = -(1 - \alpha)z^2 - \alpha$, which characterizes a local maximum. We can always transform to a frame in which $V(0) = 0$, so the velocity has a local expansion of the form of Equation (34).

To view these profiles as a local approximation, we also need to make an assumption about the relative strength and scale of variation of the magnetic field and the shear flow, since we require that $|V| = |A|$ at the boundary. When α is close to zero, the magnetic field and velocity are changing at similar rates, so the locality assumption is plausible. But when α is close to 1 or very negative, the scale heights of the flow and magnetic field are very different, so viewing these profiles as a local expansion is not as accurate.

Depending on the sign of α, the magnetic field has either two or zero nulls. When $\alpha < 0$, $A = 0$ at

$$z = \pm\sqrt{\frac{\alpha}{\alpha - 1}}.$$
(36)

When $\alpha > 0$, there are no nulls in the magnetic field, and when $\alpha = 0$, there is a single null at $z = 0$. We find that the nulls in the magnetic field are unimportant in this problem—rather, zeros of $V^2 - A^2$ are important. The eigenfunctions discussed below (see Section 6.3) show no special behavior at $A = 0$, but have sharp gradients when $|A| = |V|$. In terms of α, $|V| = |A|$ at

$$z = \pm 1,$$
(37)

$$z = \pm\frac{\alpha}{1 - \alpha}.$$
(38)

When $\alpha > 0.5$, the solutions in Equation (38) are no longer in the domain. This means that $V \leqslant A$ in the entire domain, yielding stability by Equation (24). Heuristically, when α becomes more positive, the strength of the magnetic field in the domain increases until the magnetic tension force becomes so strong that all perturbations become stable.

In the opposite limit, when α becomes very negative, the solutions in Equation (38) approach $z = \pm 1$. For arbitrarily negative α, there is still some region for which $V > A$. Tatsuno & Dorland find instability for α as small as -25, and we can prove that there is instability for all $\alpha < 0.5$ when $k^2 = 0$ using the sufficient condition for instability by Chen & Morrison described in Section 4. The explicit computation is messy, but is included in the Appendix.

The limit in which $\alpha \to -\infty$ is probably not physically relevant. As the two "almost singular" layers approach each other (see Equations (37) and (38)), there are large gradients at the boundary of the domain. In this case, the instability probably relies crucially on our choice of boundary conditions. Moreover, when stratification is included, the high field strengths and large currents corresponding to $|\alpha| \gg 1$ are destabilizing in themselves, in contrast to what we assume here. Thus, results in this limit should be viewed as proving a point about the Richardson criterion, but are not necessarily physically relevant by themselves. As we show in explicit calculations presented below, α does not need to be very negative to recover the results described in the infinitely negative case.

6.2. Effect of Stratification on Stability

Our main result is evidence for the following conjecture. There is instability as $\alpha \to -\infty$, even in the presence of arbitrarily strong density stratification, in violation of the Richardson criterion. There does not seem to be any way to prove this claim analytically, as there was in the $N^2 = 0$ case. The sufficient condition for stability presented by Chen & Morrison (1991) relies crucially on the analytic solution to the eigenvalue equation when $k^2 = 0$. When $N^2 \neq 0$, we no longer have an analytic solution to the eigenvalue equation, even when $k^2 = 0$, so there is no extension of the proof.

Given the assumptions made above, the growth rate c is a function of the following parameters: k^2, N^2, and α. We first specialize to the $k^2 = 0$ case, and then examine the more general k^2 finite case.

6.2.1. $k^2 = 0$

For this problem, the unstable area of the (N^2, α) plane is maximized for $k^2 = 0$—though this is not necessarily true in general (K68). When $k^2 = 0$, we have $c = c(N^2, \alpha)$. We have plotted contours of constant c on the N^2, α plane in Figure 1.

We find instability when $N^2 > 1/4$, violating the Richardson criterion. It seems that given an arbitrarily large value of N^2, there is a sufficiently negative value of α such that the fields are unstable. However, as mentioned in Section 6.2, the extremely negative α case is probably strongly affected by the boundary conditions.

Gravity is stabilizing: the growth rate decreases as N^2 increases. There is stability for $\alpha < 0.5$ by the same arguments as above, and as α becomes more negative, we find larger c. Although a stronger magnetic field results in a strong magnetic tension force, and the "destabilizing" region in which $|V| > |A|$ shrinks for more negative α, we nevertheless find stronger instability. We hypothesize that c increases because there is more free energy in the magnetic field as α becomes more negative and the
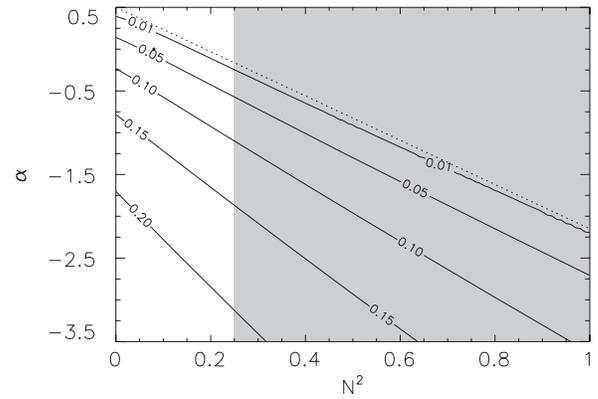
Figure 1. Contours of constant c on the α, N^2 plane. The Richardson criterion states that the shaded region is stable. The dotted line represents the $c = 0$ contour—the region below this line is unstable.

magnetic field becomes stronger. As α becomes more negative, the instability can tap more free energy from the magnetic field, and thus we find a violation of the Richardson criterion. However, note that the stronger magnetic field, and corresponding increase in magnetic free energy, is not a sufficient condition for instability, as the magnetic field is stable without the presence of shear flow.

The contours of constant c are well fit by straight lines. The equation for the boundary between the stable and unstable regimes is

$$\alpha = 0.5 - 2.65 N^2. \qquad (39)$$

Thus, for $\alpha < -0.1625$, the Richardson criterion is violated. The slopes of the contours become steeper as c increases. Although there is instability with arbitrarily large c, this does not mean the instability has an arbitrarily large growth rate. As mentioned in Section 3, the growth rate is formally zero at $k^2 = 0$. Thus, to find the growth rate, we need to understand the instability at $k^2 \neq 0$.

6.2.2. $k^2 > 0$

Although when $k^2 = 0$ there is instability for arbitrarily negative α, for every finite k, there is a cutoff α_k for which any α more negative than α_k yields stable profiles due to an insurmountable magnetic tension force. Looking at it another way, c always decreases as k increases, so for any values α and N^2 which are unstable at $k^2 = 0$, there is a k for which $c = 0$. Call this value $k_{crit}(\alpha, N^2)$. Figure 2 plots $k_{crit}(\alpha, N^2)$ as a function of α and N^2. The point (α, N^2, k) is unstable iff $k < k_{crit}(\alpha, N^2)$. Although it is possible to find instability when $k^2 > 0$ for profiles which are stable when $k^2 = 0$ (see Section 4.3), this does not seem to occur for these classes of profiles.

Figure 3 plots surfaces of constant ω in (α, N^2, k) space. The figure shows that ω is a sharply peaked function of k, and that it decreases with increasing N^2. Given $N^2, k \neq 0$, there is instability for only a finite range of α. For $N^2 \equiv 0$, our results agree with Tatsuno & Dorland (2006). For sufficiently small k, c is almost constant. Thus, the growth rate $\omega \equiv kc$ is linear in k with slope c. However, as k grows, c begins to decrease. There is a maximum growth rate defined by $d \log c / d \log k = -1$, and the growth rate goes to zero when c does. The growth rate is 1–2 orders of magnitude lower than the typical growth rates of hydrodynamic shear flow instabilities.

As k^2 increases from zero, the fluid displacement becomes more vertical. Vertical perturbations bend field lines, and are
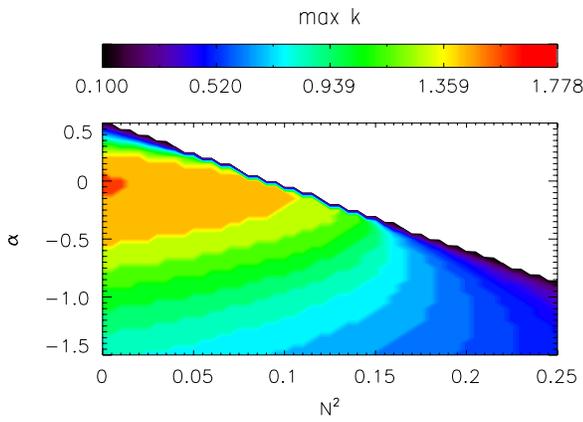
Figure 2. Largest k, denoted $k_{\rm crit}$, for each α and N^2 which is unstable. The white area is stable.

subject to a restoring magnetic tension force. Thus, it makes sense that the most unstable modes are the horizontal modes characterized by $k^2 = 0$. For some applications, such as stellar interiors (see Section 8), it is important to consider the vertical transport (of angular momentum, etc.) by these modes. In this case, the $k^2 = 0$ mode is irrelevant. One must then consider an optimization problem in which modes with too low k^2 have no vertical transport effects, whereas modes with too high k^2 are stable. This argument is only valid assuming that the nonlinear evolution is similar over a broad range of k^2. A full nonlinear simulation for various k^2 is necessary in order to understand the transport properties of these instabilities.

6.3. Eigenfunctions

We normalize the eigenfunctions as described in Section 3. The eigenfunctions all look like the example plotted in Figure 4. The most salient features are the sharp gradients at $z = \pm.47$, where $|V| = |A|$. Note that the nulls in the magnetic field at $a = \pm.69$ produce no special features.

7. CONSTANT A WITH VELOCITY PROFILES SUGGESTED BY KENT

In Section 4.3, we summarized Kent's discussion (K68) of velocity profiles which are marginally stable in the absence of a magnetic field and destabilized by a small, constant field. In this section, we generalize Kent's construction and investigate the stability of the resulting family of Kent flows.

The velocity profile is most conveniently specified by the inverse relation $z = z(V)$. Note that only invertible velocity
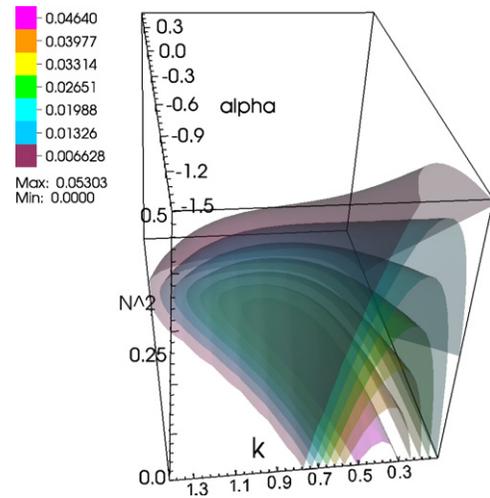
Figure 3. Surfaces of constant growth rate ω in (α, N^2, k) space. The maximum ω in this range of (α, N^2, k) is given also.

profiles, i.e., $dV/dz \neq 0$, can be specified by this inverse relation. When $k^2 = 0$ and $N^2 = 0$, we can use the instability condition by Chen & Morrison (1991) and evaluate the integral in Equation (24) in closed form. This provides a transcendental equation for the growth rate. From solving this equation numerically, it seems that there exist velocity profiles which are (marginally) stable at $A_0 = 0$, but unstable for $0 < A_0 < |V|_{\rm max}$. When we increase N^2 from zero, we always find stability when $N^2 \geqslant (\max V')^2/4$, but can find instability for all N^2 up to this limit. Our interpretation of this result is that the positive energy required to perturb a constant magnetic field triumphs over the extra freedom granted by magnetically breaking the frozen-in vorticity constraint.

7.1. $N^2 = 0$

First we consider various velocity profiles defined by $z = z(V)$ at $k^2 = 0$. Define

$$\omega(V) \equiv \frac{dz}{dV}. \tag{40}$$

We restrict ourselves to velocity profiles which are marginally stable at $A = 0$, as they seem to be maximally destabilized by magnetic fields. We will first consider velocity profiles with walls at $z = \pm z_0$, with the condition that $V(\pm z_0) = \pm 1$. This will simplify the algebra when deriving analytic stability results. We will then employ the rescaling symmetry
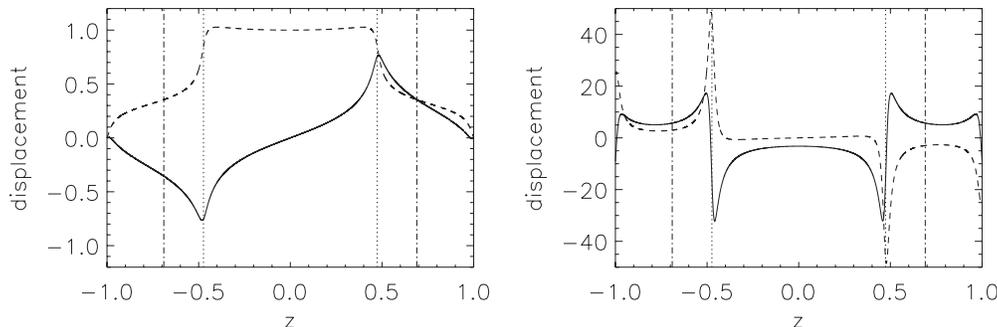
Figure 4. Vertical displacement ξ (left panel) and horizontal displacement $\xi_x = -i\xi'/k$ (right panel), where prime denotes differentiation with respect to z, eigenfunctions for $\alpha = -0.9$, $N^2 = 0.3$, and $k = 0.2$. The thick solid lines are the real part of the eigenfunctions, and the thick dashed lines are the imaginary part of the eigenfunctions. The thin vertical dotted lines are at $z = \pm.47$ where $|V| = |A|$ and the thin vertical dot-dashed lines are at $z = \pm.69$, where $A = 0$.

described in Equation (12) to present numerical results using the normalization $z_0 = 1$.

The condition for marginal stability (Kent 1968) is

$$\int_{-1}^{1} \frac{\omega(V)dV}{V} = 0,$$ (41)

where we have assumed that V ranges from -1 to $+1$ in the domain. Assuming

$$z = V + a_3 V^3 + a_5 V^5 + \cdots,$$ (42)

we have

$$\omega = 1 + 3a_3 V^2 + 5a_5 V^4 + \cdots,$$ (43)

so the marginal stability condition on the a_j's is

$$\sum_{j \geqslant 3, \text{odd}} \frac{ja_j}{j-2} = 1.$$ (44)

Our construction is a generalization of K68, who truncated the series in Equation (42) at three terms. Next we assume that there is only one inflexion point at $z = 0$. This condition implies that ω cannot have any extrema, so none of the a_j are negative. Numerical work suggests that the results discussed here hold for velocity profiles with multiple inflexion points, so by assuming only one inflexion point, we make the problem much easier, but do not qualitatively change the results.

Now we add a constant magnetic field. When $k^2 = 0$, we have that

$$\int_{-z_0}^{z_0} \frac{dz}{(V-c)^2 - A_0^2} = 0$$ (45)

implies instability with growth rate c. If we change variables to V, we find

$$\int_{-1}^{1} \frac{\omega(V)dV}{(V-c)^2 - A_0^2} = 0,$$ (46)

where $\omega(V)$ is defined as in Equation (40). We can rewrite the integral in Equation (46) as

$$\int_{-1}^{1} \frac{1}{2A_0} \omega(V)dV \left(\frac{1}{V-c-A_0} - \frac{1}{V-c+A_0} \right) = 0.$$ (47)

The two integrals have equal real parts, so all we need to calculate is

$$\Re \int_{-1}^{1} \frac{\omega(V)dV}{V-c-A_0} = 0.$$ (48)

When specifying $\omega(V)$ as a power series in odd powers of V, as in Equation (43), we can evaluate the integral by noticing that

$$\frac{1}{2} \int_{-1}^{1} \frac{V^n dV}{V-c-A_0} = \frac{c+A_0}{n-1} + \frac{(c+A_0)^3}{n-3} + \cdots + (c+A_0)^{n-1}$$
$$+ \frac{1}{2}(c+A_0)^n \left(\log(1-c-A_0) - \log(-1-c-A_0) \right),$$ (49)

and summing over each term in the power series for $\omega(V)$. This gives a transcendental condition for stability, instead of the differential condition of Equation (11).

Note that the location of the walls plays a crucial role in the equation for stability, Equation (49). Moving the walls from the z_0 where $V(z_0) = 1$ could make the marginally stable velocity profiles stable or unstable. Although we will only consider
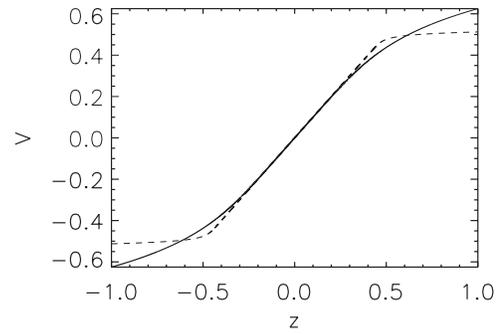
Figure 5. Velocity profile solutions of Equation (50) for $n = 5$ (solid) and $n = 41$ (dashed).

velocity profiles which are marginally stable with no magnetic field below, our results do not change qualitatively when we add a constant magnetic field to a velocity profile which is stable or unstable when $A_0 = 0$. We choose marginally stable velocity profiles because they are more clearly destabilized by magnetic fields than unstable velocity profiles, and they are more destabilized than stable velocity profiles.

For the remainder of this paper, we will normalize the problem by setting the walls at $z = \pm 1$. Under the assumptions that V has only a single inflexion point and is marginally stable at $A = 0$, we numerically find that the most unstable velocity profile at $k^2 = 0$ and $N^2 = 0$ is given by

$$z = V + \left(1 + \frac{n-2}{n} \right)^{n-1} \frac{(n-2)V^n}{n},$$ (50)

for n odd, when $n \to \infty$. In this limit, the velocity profile approaches

$$V(z) = \begin{cases} +\frac{1}{2}, & \frac{1}{2} < z < 1 \\ z, & -\frac{1}{2} < z < \frac{1}{2} \\ -\frac{1}{2}, & -1 < z < -\frac{1}{2}. \end{cases}$$ (51)

For every n odd and greater than 3, the velocity in Equation (50) is marginally stable. We plot the velocity profile for $n = 5$ and $n = 41$ in Figure 5. Note that max $V' = 1$, so the Richardson criterion states that $N^2 > 1/4$ yields stability.

For each n, we can plot c as a function of A_0 at $k^2 = 0$. Because we assumed the magnetic field is parallel to the velocity, we know there is stability when $A_0 > V_{\text{max}}$. Thus, V_{max} sets a natural scale for measuring the magnetic field strength. Figure 6 plots $c(A_0/V_{\text{max}})$ for $n = 5$ and $n = 41$. It seems that as $n \to \infty$, the maximum c approaches ≈ 0.125 for $A_0 \approx 0.65V_{\text{max}} = 0.325$.

Figure 7 shows an eigenfunction for $A_0 = 0.65V_{\text{max}} \approx 0.31$, $n = 41$. Note that it is very similar to the eigenfunction for the Tatsuno & Dorland (2006) profiles in Section 6.3.

7.2. $N^2 \neq 0$

As mentioned in Section 7.1, the velocity profiles considered here have max $V' = 1$, so the Richardson criterion states that $N^2 > 1/4$ implies stability. As n increases, the maximally unstable N^2 increases, but never seems to reach 1/4. Figure 8 shows contours of c as a function of N^2 and A_0/V_{max} for $k = 0$ and $n = 41$. Although there is instability for N^2 very close to 1/4, we find stability at $N^2 = 0.25$. It seems that the Richardson criterion is not violated when adding a constant magnetic field to this class of velocity profiles.
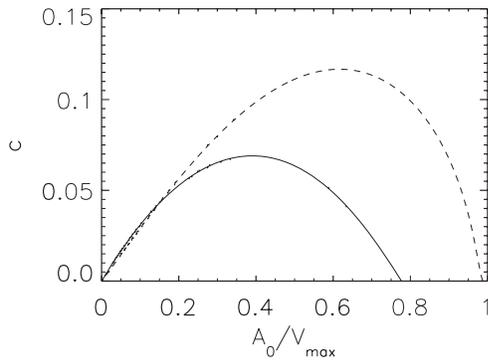
Figure 6. Imaginary part of the eigenvalue c as a function of A_0/V_{max} for the velocity profiles given by Equation (50) for $n = 5$ (solid) and $n = 41$ (dashed).

Figure 8. Contours of c as a function of N^2 and A_0/V_{max} for $k = 0$ and the velocity profile given by Equation (50) with $n = 41$. The white area is stable.
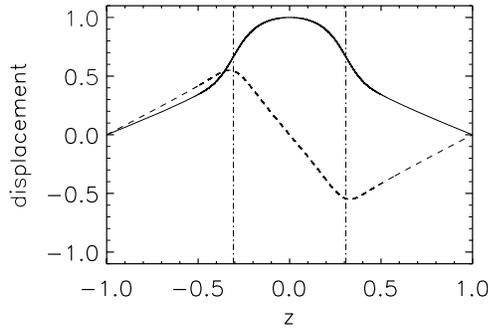
Figure 7. Eigenfunction for $A = 0.6V_{max} \approx 0.31$ and velocity given by Equation (50) for $n = 41$. The thick solid line is the real part of the eigenfunction, and the thick dashed line is the imaginary part of the eigenfunction. The vertical dotted lines denote the points where $|V| = |A|$, at $z = \pm 0.40$.

Figure 9. Eigenfunction for $A = 0.6V_{max} \approx 0.31$ and velocity given by Equation (50) for $n = 41$, with $N^2 = 0.225$, $k = 0$. The thick solid line is the real part of the eigenfunction, and the thick dotted line is the imaginary part of the eigenfunction. The vertical dot-dashed lines are at $z \approx \pm 0.31$, where $|V| = |A|$.

Figure 9 shows a typical eigenfunction. As with the velocity and magnetic field profiles considered in Section 6, there are sharp gradients when $|V| = |A|$. Unlike the eigenfunctions considered above, the real part of this eigenfunction is close to zero at the origin.

The constant magnetic field case is very different from the parabolic case because there is no violation of the Richardson Criterion. We can understand result heuristically by noting that a constant magnetic field cannot increase the free energy of the perturbation, and thus cannot render a velocity profile with $N^2 > \max V'^2/4$ unstable. Although there is no energy principle in the presence of shear flow, one can show that a sufficient condition for stability is that the energy of a perturbation is positive, i.e., $\mathbf{F}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} > 0$, where $\mathbf{F}(\boldsymbol{\xi})$ is the force operator (Frieman & Rotenberg 1960). A constant magnetic field contributes $+|\mathbf{Q}|^2$ to the energy of a perturbation, where $\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$. Thus, a constant magnetic field always increases the energy of a perturbation.

However, in Section 7.1, we describe an entire class of velocity profiles which are (marginally) stable at $A_0 = 0$, but unstable for $A_0 > 0$. Our interpretation of the destabilized is as follows. An unstable perturbation must have negative energy (Frieman & Rotenberg 1960), but this is only a necessary condition for instability. Thus, perturbations to the velocity profiles considered in Section 7.1 have negative energy, but are still stable. For a sufficiently small magnetic field ($A_0 < V_{max}$), the increase in energy of the perturbation from the magnetic field can be overcome by a negative contribution from the shear flow, so the total energy of the perturbation is negative and there could be instability.

Because the Richardson criterion can be understood from energetic arguments (see Section 4.2), one could assume that
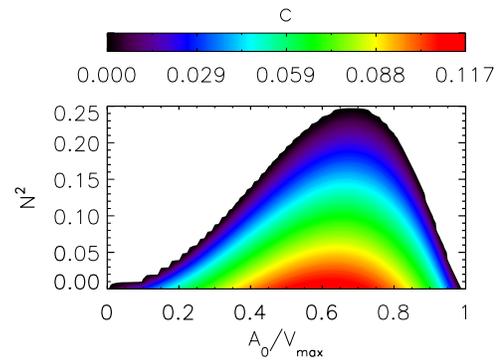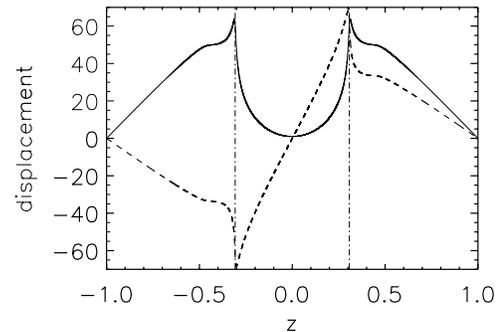
when $N^2 > V'^2/4$ in the entire domain that the energy is necessarily positive. Then the addition of a constant magnetic field only further increases the energy of the perturbation, preventing instability. This is a rather considerable assumption, so this argument is best viewed as a heuristic.

8. APPLICATION TO ASTROPHYSICAL SYSTEMS

We have studied shear flow instability in stably stratified media for flow profiles which would be stable in the absence of a magnetic field and shown that Richardson's criterion for buoyancy stabilization can be violated, provided that the magnetic field is inhomogeneous. In this section, we briefly discuss astrophysical applications.

First, some general considerations. Our analysis holds when the flow and field are perpendicular to gravity. We ignored the effect of the magnetic field on the density stratification, thereby precluding any instabilities associated with magnetic buoyancy. Thus, our work applies primarily to situations in which the field is not too strong and its scale height is not much less than the pressure scale height. Thus, although we gave an example in Section 6 of a system that can be unstable at arbitrarily large Ri, instability at large Ri required in that case that the flow be sub-Alfvénic in most of the domain and that the magnetic scale length be much less than the velocity shear length. In addition to the possible introduction of magnetic buoyancy effects, a small magnetic scale height relative to the velocity scale height requires that the magnetic Prandtl number Pm—the ratio of viscous to magnetic diffusivity—be much greater than unity, opposite to the situation in dense plasmas

such as stellar interiors. Bearing these things in mind, there is probably a practical upper limit on Ri at which magnetic fields are destabilizing according to the mechanism discussed here.

It is useful to cast Ri in a form which allows its magnitude to be estimated. We introduce a buoyancy parameter f_{bu} in terms of which N^2 can be written in terms of the local gravity and pressure scale height as

$$N^2 = f_{bu} \frac{g}{H_\rho}, \qquad (52)$$

where g and H_ρ are the local gravity and density scale height, respectively; in the Boussinesq approximation, $f_{bu} = 1$. Specializing to the case that V is a rotational velocity, we introduce the velocity scale height H_v by $V' = V/H_v$ and a breakup parameter f_{br} by

$$|V'|^2 = f_{br} \frac{rg}{H_v^2}, \qquad (53)$$

where r is the distance from the rotation axis. Using Equations (52) and (53), Ri can be written as

$$\mathrm{Ri} = \frac{f_{bu}}{f_{br}} \frac{H_v}{H_\rho} \frac{H_v}{r}. \qquad (54)$$

In stably stratified systems with uniform composition, f_{bu} is generally $\mathcal{O}(1)$, while a molecular weight gradient can render $f_{bu} \gg 1$. Except for systems rotating near breakup, $f_{br} \ll 1$. Typically, H_v exceeds the geometric width of a shear layer because V changes by only a fraction of itself. Thus, although the second and third ratios on the rhs of Equation (54) are below unity, they are generally not enough to offset f_{bu}/f_{br}, and Ri $\gg 1$. One exception to these considerations occurs near the boundaries of convection zones, where N^2 crosses through zero. Thus, a thin layer on the stably stratified side of the boundary could be magnetically destabilized even if Ri $> 1/4$.

The expectation that Ri $\gg 1$ in the stably stratified portions of stellar interiors is borne out by examination of stellar models. First, we consider the Sun. Helioseismology has revealed a thin shear layer, known as the tachocline, below the base of the solar convection zone, which is thought to lie at $0.713\,R_\odot$ (see Gough 2007 for a review). If we take N^2 at $0.700\,R_\odot$ from Gough and V' from Schatzman et al. (2000), we find that at the equator Ri $= 6400$ and $f_{bu} \sim 10^{-2}$. In other words, even very close to the base of the convection zone Ri is quite large, and increases with depth from the value given here.

We also evaluated Ri in an evolutionary sequence of models of massive, rotating stars generously provided to us by G. Meynet. The initial mass is 20 $M_\odot$ (which decreases due to mass loss) and the initial surface rotation period is about 1.2 days. When the star first reaches the main sequence, the core is convective and the envelope is radiative. As hydrogen is exhausted in the core, strong nonhomologous contraction spins up the core and creates strong shear layers, which tends to reduce Ri. At the same time, steep negative molecular weight gradients increase f_{bu}. We find that in the bulk of the interior, Ri is between 10^2 and 10^6. In the models, the boundaries of convection zones (which form in association with shell burning) actually show spikes in Ri. This is because Ω is set to a constant in convection zones, due to efficient turbulent mixing. Thus, although there is probably a thin layer in which Ri drops to small values, it cannot be evaluated from these models.

These estimates suggest that destabilization of stellar rotation profiles by weak magnetic fields is likely to occur only in thin layers outside convection zones. However, the tendency for such fields to destabilize a system may be important even when physical processes neglected by our analysis are included. Chief among them is thermal diffusion, which can suppress the stabilizing effects of buoyancy (Zahn 1974) and leads to a larger critical Ri to guarantee stabilization. Whether this carries over our analysis is a topic for future study.

The instability could conceivably also operate on poloidal flows. However, because such flows are generally slow compared with rotation, their Ri tends to be even larger than Ri for rotation. And because rotational shear tends to make the magnetic field predominantly toroidal, magnetic effects on the stability of poloidal flow are probably weak.

Similar considerations hold for accretion disks. The vertical shear in a Keplerian disk of thickness H is smaller than the radial shear by a factor of H/r. If the radial inflow velocity is a function of height, its shear could be large, but the magnetic field is expected to be predominantly toroidal. Therefore, this instability is probably not critically important for either rotation or radial flow in disks.

9. CONCLUSION

Turbulence is a key ingredient in the transport of chemical species, entropy, angular momentum, and magnetic flux in astrophysical settings. Shear flows, which are driven almost ubiquitously in nature, can become turbulent through instability.

In this paper, we have considered ideal instabilities of magnetized shear flows in stably stratified systems. In the absence of magnetic fields, the Richardson criterion provides a necessary condition for instability based on comparing the kinetic energy released by vertical interchange of fluid elements to the potential energy required to displace them. The Richardson criterion is often assumed to set the ideal stability boundary for shear flow instabilities in stratified media such as stars and accretion disks. The main result of this paper is that the Richardson criterion is no longer valid when inhomogeneous magnetic fields are included: because such fields carry free energy, buoyancy forces must be stronger to stabilize the system. We have provided an example by adding density stratification to the fields described by Tatsuno & Dorland (2006). These fields can be viewed as a local approximation of any shear flow in the presence of a magnetic extremum. The system has the interesting property that the flow is neutrally stable in the absence of the magnetic field, but unstable in its presence. Solving the eigenvalue problem in Equation (11), we find unstable modes for arbitrarily large N^2, provided that the magnetic field is sufficiently strong. Even for magnetic fields yielding Alfvén velocities comparable to flow velocities, we find violation of the Richardson criterion. Thus, when considering the ideal stability of a plasma shear flow in a stratified medium, it is not sufficient to consider the Richardson criterion.

We were unable to find an example in which a *constant* magnetic field leads to violation of the Richardson criterion. We extended and analyzed a class of velocity profiles considered by Kent (1968), which were shown to be destabilized by a constant magnetic field. Although we were able to destabilize the flows when $N^2 = 0$, and the fastest growing modes have moderately strong magnetic fields, when $N^2 > V'^2/4$, we always found stability. We provided two heuristics for understanding the destabilization due to magnetic fields. An inhomogeneous magnetic field provides a free energy source which can be tapped by an instability. Thus, while a homogeneous magnetic field can be destabilizing because vorticity is no longer frozen

into the flow, allowing new unstable plasma motions, only an inhomogeneous field can provide the source of energy needed to violate Richardson's criterion.

We briefly applied our results to the solar tachocline and to high mass, rapidly rotating stars. In the bulk of the tachocline, Ri is very large because the Sun rotates slowly. Very near the boundary of the convection zone, Ri drops because N^2 is passing through zero. A similar situation holds, for different reason, in high mass stars. Although these stars rotate rapidly, the regions of strong shear coincide with regions of strong, stabilizing, molecular weight gradient. This keeps Ri large, except near convection zone boundaries. Thus, in stars, the destabilization of stratified shear flow by magnetic fields is most likely to occur in thin regions on the stable side of convection zone boundaries. If the weakening of buoyancy by thermal diffusion destabilizes magnetized flow in the same way as unmagnetized flow, the unstable region could be much larger, however.

Our two-dimensional slab model is not a realistic geometry for many applications. The introduction of additional terms, such as curvature terms from toroidal geometry or the centrifugal force for rotation, probably changes our results quantitatively, but not qualitatively. The Boussinesq approximation could also be relaxed to allow more realistic density profiles and other physics. Inclusion of diffusive effects would allow us to consider non-ideal instabilities, including the secular shear instability. For many applications, the nonlinear phase and saturation of these instabilities is also important for determining effects such as angular momentum transport. These considerations should be investigated further to better understand the nature of magneto-shear instabilities in a stratified medium.

APPENDIX

INSTABILITY OF $V = z$, $A = (1 - \alpha)z^2 + \alpha$ WHEN $\alpha < 0.5$

We will prove that the velocity and magnetic field profiles considered in Section 6, $V = z$, $A = (1 - \alpha)z^2 + \alpha$, are unstable when $\alpha < 0.5$. In Section 4.3, we described the following sufficient condition for instability at $k^2 = 0$ by Chen & Morrison (1991; Equation (24)): If

$$\int_{-1}^{1} \frac{1}{(V - i\epsilon)^2 - A^2} > 0 \tag{A1}$$

as $\epsilon \to 0$, then there is instability. We can factor the denominator to get

$$\frac{1}{2} \int_{-1}^{+1} \frac{dz}{A(V - i\epsilon - A)} - \frac{1}{2} \int_{-1}^{+1} \frac{dz}{A(V - i\epsilon + A)}. \tag{A2}$$

Let us examine how these two integrals are related. Define $u = -z$. Then

$$-\frac{1}{2} \int_{-1}^{+1} \frac{dz}{A(z)(V(z) - i\epsilon + A(z))}$$
$$= \frac{1}{2} \int_{+1}^{-1} \frac{du}{A(z)(V(z) - i\epsilon + A(z))}$$
$$= -\frac{1}{2} \int_{-1}^{+1} \frac{du}{A(u)(-V(u) - i\epsilon + A(u))}$$
$$= \frac{1}{2} \int_{-1}^{+1} \frac{du}{A(u)(V(u) + i\epsilon - A(u))}, \tag{A3}$$

which has the same real part as the first integral, but opposite imaginary part. Thus, we need only check that

$$\Re \int_{-1}^{+1} \frac{dz}{A(V - i\epsilon - A)} > 0 \tag{A4}$$

as $\epsilon \to 0$ to prove instability. Integrals of this form can be evaluated in a closed form, but must first be factored. To simplify the algebra, we reduce the degree of the polynomial in the denominator through partial fractions.

$$\Re \int_{-1}^{+1} \frac{dz}{A(V - i\epsilon - A)}$$
$$= \Re \int_{-1}^{+1} \frac{dz}{A(V - i\epsilon)} + \Re \int_{-1}^{+1} \frac{dz}{(V - i\epsilon)(V - i\epsilon - A)}. \tag{A5}$$

The first integral gives no contribution because multiplying by $V + i\epsilon$ in the numerator, and denominator shows that the real part is odd and integrates to zero. Thus, we need only evaluate the second integral.

We can integrate the remaining part by brute force, i.e., using Mathematica. Assuming $\epsilon > 0$, Mathematica gives

$$\int \frac{dz}{(V - i\epsilon)(V - i\epsilon - A)}$$
$$= -\frac{1}{4(\alpha - \epsilon^2 + \alpha\epsilon^2)} \left[-4i \arctan\left(\frac{\epsilon}{z}\right) \right.$$
$$- \log\left(\epsilon^2 + (-1 + z)^2 z^2 - 2\alpha(-1 + z)^2 z(1 + z) + \alpha^2(-1 + z^2)^2\right)$$
$$+ \frac{4(1 - 2i(1 - \alpha)\epsilon)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i + \epsilon)}}$$
$$\times \arctan\left(\frac{-1 + 2(1 - \alpha)z}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i + \epsilon)}}\right)$$
$$+ 2i \arctan\left(\frac{(-1 + z)(z - \alpha z - \alpha)}{\epsilon}\right) + 2 \log\left(\epsilon^2 + z^2\right) \right]. \tag{A6}$$

Note that the prefactor has the opposite sign as α. The term on the first line is imaginary, so we do not need to consider it. In the logarithm on the second line, the third and fourth terms which are 0 at $z = \pm 1$. On the last line, the first term is imaginary and the second term is even, so neither contribute to the integral. Thus, if

$$-\frac{1}{4(\alpha - \epsilon^2 + \alpha\epsilon^2)} \left[\log\left(\frac{\epsilon^2 + 4}{\epsilon^2}\right) \right.$$
$$+ \Re \frac{4(1 - 2i(1 - \alpha)\epsilon)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i + \epsilon)}}$$
$$\times \arctan\left(\frac{-1 + 2(1 - \alpha)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i + \epsilon)}}\right)$$

$$- \Re \frac{4(1 - 2i(1-\alpha)\epsilon)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i+\epsilon)}}$$

$$\times \arctan\left(\frac{-1 - 2(1-\alpha)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i+\epsilon)}}\right)\Bigg] > 0 \quad (A7)$$

for a particular α as $\epsilon \to 0$, then the profiles for that α are unstable. The ϵ for which the rhs of Equation (A7) equals zero is the growth rate of the instability. Thus, this relation gives a transcendental equation for the growth rate, which is significantly easier to solve than the differential eigenvalue problem given in Section 3.

As $\epsilon \to 0$, the logarithm term diverges and is positive. However, when $z = +1$, the arctan term also diverges, approaching $-i\infty$, meaning that the entire term gives a negative divergent contribution. We need to see which diverges faster. The argument of the $z = +1$ arctan term is

$$\frac{1 - 2\alpha}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i+\epsilon)}}$$

$$= -i \frac{1 - 2\alpha}{\sqrt{4\alpha^2 - 4\alpha + 1 - 4i\epsilon(1-\alpha)}}$$

$$= -i \left(1 - \frac{4i\epsilon(1-\alpha)}{4\alpha^2 - 4\alpha + 1}\right)^{-1/2}$$

$$\approx -i \left(1 + \frac{1}{2}\frac{4i\epsilon(1-\alpha)}{(1-2\alpha)^2}\right). \quad (A8)$$

In general, $\arctan(z)$ is given by

$$\arctan(z) = i\frac{1}{2}\left(\log(1 - iz) - \log(1 + iz)\right). \quad (A9)$$

The divergent part for us is the first term, so

$$\arctan\left(\frac{-1 - 2(1-\alpha)}{\sqrt{-1 - 4\alpha^2 + 4i\epsilon - 4i\alpha(i+\epsilon)}}\right)$$

$$\approx i\frac{1}{2}\log\left(-\frac{2i\epsilon(1-\alpha)}{(1-2\alpha)^2}\right). \quad (A10)$$

If we neglect the ϵ terms which are not in the divergence, we find that the coefficient of the $\log(\epsilon)$ term is $-2/(1-2\alpha)$. Thus, only considering the terms in Equation (A7) which are divergent as $\epsilon \to 0$, and taking $\epsilon = 0$ except for in the divergence, we are left with

$$-\frac{1}{4\alpha}\left(2\log(\epsilon) - \frac{2}{1-2\alpha}\log(\epsilon)\right). \quad (A11)$$

When $\alpha < 0$, we have that $-1/4\alpha > 0$, and the first $\log(\epsilon)$ term dominates, so the whole quantity is positive. Thus, we have

proven that there is instability for $\alpha < 0$. When $0.5 > \alpha > 0$, we have $-1/4\alpha < 0$, but the second logarithm term dominates and is negative, again yielding instability. However, when $\alpha > 0.5$, both divergent terms become positive, but $-1/4\alpha < 0$, so the quantity is negative as $\epsilon \to 0$, and the profiles are stable. In order to show instability at $\alpha = 0$, we would need to retain more terms in our perturbative expansion in ϵ.

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