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Surface trapping and leakage of low-frequency g modes in rotating early-type stars – I. Qualitative analysis

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ABSTRACT

A qualitative study of the surface trapping of low-frequency non-radial g modes in rotating early-type stars is undertaken within the Cowling, adiabatic and traditional approximations. A dispersion relation describing the local character of waves in a rotating star is derived; this dispersion relation is then used to construct propagation diagrams for a 7-$M_\odot$ stellar model, which show the location and extent of wave trapping zones inside the star. It is demonstrated that, at frequencies below a cut-off, waves cannot be fully trapped within the star, and will leak through the surface. Expressions for the cut-off frequency are derived in both the non-rotating and rotating cases; it is found from these expressions that the cut-off frequency increases with the rotation rate for all but prograde sectoral modes.

While waves below the cut-off cannot be reflected at the stellar surface, the presence of a sub-surface convective region in the stellar model, owing to He II ionization, means that they can become partially trapped within the star. The energy leakage associated with such waves, which are assigned the moniker *virtual modes* owing to their discrete eigen-frequencies, means that stability analyses which disregard their existence (by assuming perfect reflection at the stellar surface) may be in error.

The results are of possible relevance to the 53 Per and SPB classes of variable star, which exhibit pulsation frequencies of the same order of magnitude as the cut-off frequencies found for the stellar model. It is suggested that observations either of an upper limit on variability periods (corresponding to the cut-off), or of line-profile variations owing to virtual modes, may permit asteroseismological studies of the outer layers of these systems.

Key words: stars: early-type – stars: oscillations – stars: rotation.

1 INTRODUCTION

The self-excitation of global non-radial pulsation modes in a star is a prime example of positive feedback, whereby small oscillatory perturbations grow in amplitude via the efficient conversion of heat into vibrational energy by a suitable driving mechanism (see, e.g., Unno et al. 1989 for a comprehensive review of the topic). A fundamental ingredient in the feedback loop is that the oscillations must be trapped in some part of the stellar interior, so that energy does not leak from the system faster than it can be generated. Such trapping can occur when a pair of evanescent regions, where traveling waves cannot be supported, enclose a propagative region; waves are repeatedly reflected at the two evanescent boundaries, and the resulting superposition leads to a standing wave of the normal-mode type.

For waves excited in stellar envelopes, it is common for the surface layers to serve as one of the evanescent regions required for the formation of a trapping zone. Ando & Osaki (1975) demonstrated that such a situation occurs in the Sun, where low-order p modes are trapped beneath the photosphere, supporting a model first put forward by Ulrich (1970) to explain the five-minute solar oscillation Leighton, Noyes & Simon (1962). However, the trapping is only effective for modes with frequencies below some cut-off; higher frequency modes cannot be reflected at the photosphere, and will leak through the stellar surface. This issue was addressed in detail by Ando & Osaki (1977), who found that, although leakage does occur through the solar photosphere at frequencies above the cut-off, some waves can subsequently be reflected at the chromosphere-corona interface, and standing waves are able to form. More recently, Balmforth & Gough (1990) suggested that such coronal reflection can explain apparent observations of high-frequency chromospheric standing waves (Fleck & Deubner 1989), although debate concerning this interpretation still continues (Kumar et al. 1994; Dzhalilov & Staude 1995; Jefferies 1998).

Pulsation in massive, early stars (types O and B) is qualitatively quite different from the solar case, owing to the gross structural differences between the two stellar classes. However, it is still

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subject to the same wave trapping requirements, since the underlying physics remains the same. Shibahashi & Osaki (1976), in their study of g modes trapped within the hydrogen-burning shell of evolved massive stars, found that high-frequency (low-order) modes can tunnel through an evanescent region separating core and envelope, and thence escape from the star. A complementary situation was discussed by Osaki (1977) when studying pulsation in Cepheid-type stars; non-radial p modes trapped within the envelope were able to tunnel through an evanescent region into the core, where they were damped rapidly without reflection at the centre.

In both cases, the appropriate region of the star was modelled as an isolating oscillating unit with the inclusion of wave leakage at one boundary. The leakage was found to stabilize some modes which would otherwise have been self-excited, as a result of the associated loss of vibrational energy from the star. Shibahashi (1979) analysed wave trapping in an idealized stellar model (corresponding to an evolved massive star) using an asymptotic method, and discussed in some depth these two cases; in addition, he considered the situation where low-frequency (high-order) g modes are able to tunnel through an evanescent region in the envelope and thence escape through the stellar surface.

More recently, however, relatively little attention has been shown regarding wave trapping issues at the surface of early-type stars; in particular, stability analyses (Cox et al. 1992; Kiriakidis, El Eid & Glatzel 1992; Dziembowski & Pamyatnykh 1993; Gautschy & Saio 1993), based on the new opacity calculations of Rogers & Iglesias (1992) and Seaton (1993), have assumed that the Lagrangian pressure perturbation δp tends to zero or some limiting value at the stellar surface. Such an assumption corresponds to the *ab initio* condition that waves incident from the interior are totally reflected at the stellar surface; the possibility of leakage is thereby disregarded, and no consideration of trapping issues is undertaken.

This is the first in a short series of papers studying the surface trapping of low-frequency g modes in early-type stars, in an attempt to re-open discussion of, and investigation into, this important area. Much of the work is conceptually developed from that of Ando & Osaki (1975); however, in light of recent research into the influence of rotation on low-frequency modes (Lee & Saio 1990, 1997; Bildsten, Ushomirsky & Cutler 1996), and owing to the fact that significant rotation appears to be commonplace in O- and B-star populations (Howarth et al. 1997), the theory is updated to include rotational effects.

The current paper serves as a introduction, covering the more qualitative, general aspects of the study; subsequent papers will investigate various issues arising from this paper in greater depth. The following section reviews the pulsation equations appropriate for low-frequency g modes in rotating stars, whilst Section 3 derives the dispersion relation corresponding to these equations. The trapping of waves described by this dispersion relation is examined in Section 4 with the aid of propagation diagrams, and the effect of rotation on the eigenfrequencies of individual modes is discussed in Section 5. The findings are discussed in Section 6, and summarized in Section 7.

2 PULSATION EQUATIONS

The dynamics of pulsation in a rotating star differ from the non-rotating case as a result of the influence of the fictitious Coriolis and centrifugal forces, which arise as a consequence of the non-inertial nature of a rotating frame of reference. The centrifugal force breaks the equilibrium symmetry of the star, so that the level (equipotential) surfaces become oblate spheroids rather than the usual concentric spheres. Such a change in stellar configuration will manifest itself implicitly in the pulsation equations, through modifications to the equilibrium variables of state. However, in the case of uniform (solid-body) rotation, no *explicit* modification of the pulsation equations occurs as a result of this centrifugal distortion (Unno et al. 1989). In contrast, the Coriolis force enters the pulsation equations explicitly, through the introduction of a velocity-dependent term in the hydrodynamical momentum equation. This term can lead to the significant modification of individual pulsation modes, and is also responsible for the existence of new classes of wave-like solutions (Longuet-Higgins 1968) which are not found in non-rotating systems.

Simultaneous treatment of both forces within a pulsation framework is fraught with difficulty. Some progress towards this goal has been made (e.g. Lee 1993; Lee & Baraffe 1995), but attempts remain frustrated by the fact that the centrifugal distortion cannot really be considered as an a posteriori modification to the structure of a given star, but must be treated self-consistently with the evolution of the star (see, e.g., Meynet & Maeder 1997). However, in a number of limiting cases, certain approximations can be made which simplify the problem significantly. In the case of the low-frequency modes, the Coriolis force will dominate the centrifugal force, and the effects of the latter on the equilibrium configuration may be disregarded if the rotation is not too severe. Furthermore, the so-called 'traditional approximation' (Eckart 1960) may be employed, whereby the horizontal component of the angular frequency vector of rotation $\mathbf{\Omega}$ is neglected. This approximation is most appropriate for low-frequency pulsation modes in the outer regions of a star (Unno et al. 1989), and therefore can be considered useful in the present study.

In combination with the Cowling (1941) and adiabatic approximations, where the perturbations to the gravitational potential and specific entropy, respectively, are neglected, the traditional approximation renders the pulsation equations separable in the spherical polar co-ordinates (r, θ, ϕ). Solutions for the dependent variables ξ_r and p', the radial fluid displacement and Eulerian pressure perturbation, respectively, may then be written in the form (Lee & Saio 1997)

$$\xi_r = \xi_r(r)\Theta_l^m(\mu;\nu)\exp[\mathrm{i}(m\phi + \omega t)], \tag{1}$$

$$p' = p'(r)\Theta_l^m(\mu;\nu)\exp[\mathrm{i}(m\phi + \omega t)], \tag{2}$$

where $\mu \equiv \cos\theta$ is the normalized latitudinal distance from the equatorial plane, ω is the pulsation frequency in the co-rotating reference frame, and $\Theta_l^m(\mu;\nu)$ is a Hough function (Bildsten et al. 1996; Lee & Saio 1997). These Hough functions are the eigensolutions of Laplace's tidal equation (Longuet-Higgins 1968), and form a one-parameter family in $\nu \equiv 2\Omega/\omega$, where $\Omega \equiv |\mathbf{\Omega}|$ is the angular frequency of rotation. The integer indices l and m, with $l \geqslant 0$ and $|m| \leqslant l$, correspond to the harmonic degree and azimuthal order, respectively, of the associated Legendre polynomials $P_l^m(\mu)$ (Abramowitz & Stegun 1964) to which the Hough functions reduce in the non-rotating limit, so that $\Theta_l^m(\mu;0) \equiv P_l^m(\mu)$. This indexing scheme, based on the one adopted by Lee & Saio (1990), is less general than that of Lee & Saio (1997), in that it does not encompass the Hough functions corresponding to Rossby and oscillatory convective modes (which do not have non-rotating counterparts); however, such modes are not considered herein, and the current scheme is sufficient. Note that ω is considered to be positive throughout the following discussion, and, therefore, prograde and retrograde modes correspond to negative and positive values of m, respectively.

The radial dependence of the solutions (1–2) is described by the eigenfunctions $\xi_r(r)$ and $p'(r)$, which are governed by a pair of coupled first-order differential equations. In order to facilitate subsequent manipulation, it is useful to write these equations in the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r) - \frac{g}{c_\mathrm{s}^2}\xi_r = \frac{1}{\omega^2 c_\mathrm{s}^2}\left(\frac{\lambda_{\mathrm{lm}}^2 c_\mathrm{s}^2}{r^2} - \omega^2\right)\frac{p'}{\rho} \tag{3}$$

and

$$\frac{1}{\rho}\frac{\mathrm{d}p'}{\mathrm{d}r} + \frac{g}{c_\mathrm{s}^2}\frac{p'}{\rho} = (\omega^2 - N^2)\xi_r, \tag{4}$$

where ρ, c_s, g and N are the local equilibrium values of the density, adiabatic sound speed, gravitational acceleration and Brunt–Väisälä frequency, respectively. Note that ξ_r and p' are now taken to be functions of r alone in both these and subsequent equations, unless explicitly stated.

The quantity λ_{lm} appearing in equation (3), which arises as separation constant when solutions of the form (1–2) are sought, is the eigenvalue of Laplace's tidal equation corresponding to the appropriate Hough function $\Theta_l^m(\mu;\nu)$. In the limit $\nu = 0$, this eigenvalue is equal to $l(l+1)$, and equations (3–4) are then identical to those appropriate for a non-rotating star (e.g. Unno et al. 1989, section 15.1). The utility of the traditional approximation thus lies in the fact that much of the formalism of the non-rotating case may also be applied to rotating stars with the simple replacement of $l(l+1)$ by λ_{lm}, a result first found by Lee & Saio (1987).

Global solution of equations (3–4) must typically be approached numerically; however, an examination of the local character of the solutions suffices in the present qualitative context. This character is governed by the dispersion relation applicable to the equations, discussed in the following section.

3 DISPERSION RELATION

To derive a local dispersion relation for the pulsation equations (3–4), it is useful first to place the equations in a canonical form similar to that introduced by Osaki (1975) for the non-rotating case. By defining the two new eigenfunctions,

$$\tilde{\xi} = r^2\xi_r \exp\left(-\int_0^r \frac{g}{c_\mathrm{s}^2}\,\mathrm{d}r\right), \tag{5}$$

$$\tilde{\eta} = \frac{p'}{\rho}\exp\left(-\int_0^r \frac{N^2}{g}\,\mathrm{d}r\right), \tag{6}$$

the left-hand sides of both pulsation equations may be written as a single derivative, and the canonical form is found as

$$\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}r} = h(r)\frac{r^2}{c_\mathrm{s}^2\omega^2}\left(\frac{\lambda_{lm}c_\mathrm{s}^2}{r^2} - \omega^2\right)\tilde{\eta}, \tag{7}$$

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}r} = \frac{1}{r^2 h(r)}(\omega^2 - N^2)\tilde{\xi}, \tag{8}$$

where

$$h(r) = \exp\left[\int_0^r \left(\frac{N^2}{g} - \frac{g}{c_\mathrm{s}^2}\right)\mathrm{d}r\right]. \tag{9}$$

Note that $h(r)$ is always positive, so that the original eigenfunctions ξ_r and p' everywhere share the same sign as $\tilde{\xi}$ and $\tilde{\eta}$, respectively.

Qualitative solution of these canonical equations is accomplished using the same method as Osaki (1975), namely, by assuming that the coefficients on the right-hand sides are independent of r. Such an assumption will be valid if the characteristic variation scale of the solutions is much smaller than that of the coefficients. Then, local solutions of the form

$$\tilde{\xi}, \tilde{\eta} \sim \exp(\mathrm{i}k_r r), \tag{10}$$

lead to a dispersion relation for the radial wavenumber k_r,

$$k_r^2 = \frac{1}{c_\mathrm{s}^2\omega^2}\left(\frac{\lambda_{lm}c_\mathrm{s}^2}{r^2} - \omega^2\right)(N^2 - \omega^2). \tag{11}$$

By introducing the effective transverse wavenumber k_tr, defined by Bildsten et al. (1996) as

$$k_\mathrm{tr}^2 = \frac{\lambda_{lm}}{r^2}, \tag{12}$$

the dispersion relation may be re-written in the more useful form

$$k_r^2 c_\mathrm{s}^2\omega^2 = (k_\mathrm{tr}^2 c_\mathrm{s}^2 - \omega^2)(N^2 - \omega^2), \tag{13}$$

The value of k_r for given ω and r, calculated using this expression, determines the local character of waves at the appropriate frequency and location within the star. Inspection of equation (10) shows that real values ($k_r^2 > 0$) correspond to propagative regions, where the waves oscillate spatially, whilst imaginary values ($k_r^2 < 0$) correspond to evanescent regions, where the waves grow or decay exponentially in amplitude. The $k_r^2 = 0$ curves in the (r, ω^2) plane, defined by the roots of the right-hand side of the dispersion relation, separate these two types of region, and therefore correspond to the reflective boundaries discussed in the introduction. These boundaries, of fundamental importance when trapping zones are considered, are examined in the following section with the aid of propagation diagrams. Note that this $k_r^2 = 0$ definition of the reflective boundaries formally violates the assumption used previously to derive the solutions (10); however, this violation will have little effect on the positions of the boundaries, and is not important at a qualitative level.

The remainder of this section is left to a discussion of the effective transverse wavenumber k_tr, since, as will be demonstrated subsequently, this quantity can be pivotal in determining the trapping conditions at the stellar surface. In the case of plane waves in an infinite, plane-parallel, stratified medium, k_tr may be regarded as a free parameter; however, in the case of a spherical configuration, its is constrained to assume values permitted by equation (12). These constraints arise from to transverse boundary conditions applicable to waves propagating in horizontal (i.e., non-radial) directions; in a non-rotating star, they are equivalent to the requirement that solutions are invariant under the periodic transformations $\theta \to \theta + 2\pi$ and $\phi \to \phi + 2\pi$, and lead to the familiar result (e.g. Unno et al. 1989)

$$k_\mathrm{tr}^2 = \frac{l(l+1)}{r^2} \qquad (\nu = 0). \tag{14}$$

When significant rotation is introduced, both of these requirements still hold, but an additional constraint in θ is introduced as a consequence of the variation of the Coriolis force with latitude. Such variation means that, for $\nu > 1$, waves near the equator which are propagating in the latitudinal direction become evanescent when $|\mu| > 1/\nu$; subsequent reflections lead to the trapping of these waves within the so-called 'equatorial waveguide' (Gill 1982). The resulting horizontally-standing waves, whose angular dependence is described by the Hough functions $\Theta_l^m(\mu;\nu)$, are oscillatory in latitude between the waveguide boundaries at $\mu = \pm 1/\nu$, and evanescent elsewhere. With increasing ν, these boundaries converge towards the equator; for significant rotation, the constraints on $k_{\rm tr}$ therefore become dominated by the approximate requirement that an integer number of half-wavelengths in latitude fit between the waveguide boundaries. This requirement is manifest in the asymptotic expression for λ_{lm} found by Bildsten et al. (1996), which, when substituted into equation (12), gives

$$k_{\rm tr}^2 = \frac{(2l_\mu - 1)^2 \nu^2}{r^2} \qquad (\nu \gg 1), \tag{15}$$

where l_μ is the number of latitudinal nodes exhibited by the appropriate Hough function between the waveguide boundaries. This latter quantity is independent of ν for prograde ($m > 0$) and zonal (m=0) modes, whilst it increments by 2 as ν is increased beyond unity for retrograde ($m < 0$) modes, due to the introduction of an additional pair of latitudinal nodes at $\nu = 1$ (Lee & Saio 1990; Lee & Saio 1997). Furthermore, $l_\mu = l - |m|$ for $\nu = 0$, since the associated Legendre polynomials $P_l^m(\mu)$ exhibit $l - |m|$ zeroes over $-1 < \mu < 1$, and $\Theta_l^m(\mu;0) \equiv P_l^m(\mu)$. These properties mean that l_μ in the above asymptotic expression (15) may be written in terms of l and m as

$$k_{\rm tr}^2 = \frac{(2l - 2|m| \pm 1)^2 \nu^2}{r^2} \qquad (\nu \gg 1), \tag{16}$$

the plus sign being chosen for retrograde modes ($m > 0$), and the minus sign for prograde and zonal modes ($m \leqslant 0$). A comparison of this result with equation (14) indicates that the permitted values of $k_{\rm tr}$ for $\nu \gg 1$ can greatly exceed the corresponding ones in the non-rotating case, especially for small $|m|$.

The notable exception to this discussion is the case of the prograde sectoral modes ($m = -l$), which in the limit $\nu \gg 1$ are transformed into equatorially-trapped Kelvin waves. Such Kelvin waves have a exponential latitudinal dependence at small μ described by (Gill 1982)

$$\xi_r, p' \sim \exp\left(-\frac{\Omega^2 \mu^2 r^2}{c_{\rm s}^2}\right), \tag{17}$$

indicating that they should be considered evanescent in the latitudinal direction even at the equator. Therefore, the constraints on $k_{\rm tr}$ are dominated by the periodic boundary condition in ϕ; the $\exp({\rm i}m\phi)$ azimuthal dependence of the solutions (1–2) then gives the transverse wavenumber for prograde sectoral modes as

$$k_{\rm tr}^2 = \frac{m^2}{r^2} \qquad (\nu \gg 1, m = -l), \tag{18}$$

which can also be derived using the asymptotic expression for λ_{lm} found by Bildsten et al. (1996) for these modes.

4 WAVE TRAPPING AND LEAKAGE

As was demonstrated in the preceding section, the character of waves within a star is determined by the local radial wavenumber, so that positive and negative values of k_r^2 can be identified with propagative and evanescent regions, respectively. An indispensable diagnostic tool for visualizing the location and extent of these regions, over a range of frequencies, is the propagation diagram introduced by Scuflaire (1974), in which the (r, ω^2) plane is divided into zones over which the sign of k_r^2 is constant.

Fig. 1 shows the propagation diagram for $l = 4$ modes in a typical (non-rotating) early-type star; the logarithm of the temperature T has been adopted as the abscissa, rather than the radius, to emphasize the outer regions of the star. Regions in the $(\log T, \omega^2)$ plane where waves are propagative ($k_r^2 > 0$) are hatched, whilst evanescent regions ($k_r^2 < 0$) are blank. Values for the Brunt–Väisälä frequency N and adiabatic sound speed c_s throughout the star, required for the evaluating k_r^2 using the dispersion relation (13), have been taken from a 7-$M_\odot$ ZAMS stellar model, calculated by Loeffler (private communication), the parameters of which are summarized in Table 1; the model extends out to the photosphere at optical depth $\tau = 2/3$, where the temperature $T \equiv T_{\rm eff} = 21\,000$ K corresponds to an early-B spectral type. Equation (14), which is appropriate in the non-rotating case, has been used to calculate $k_{\rm tr}$.

In this figure, the type of hatching used to show propagative regions delineates between waves with p- and g-mode characters, the former occurring when both parenthetical terms on the right hand side of the dispersion relation (13) are negative, and the latter when both terms are positive. The division of the diagram into relatively distinct p- and g-mode propagation regions is characteristic of early-type stars. This division arises owing to the fact that $k_{\rm tr}^2 c_{\rm s}^2$ diverges at the origin (owing to the $1/r$ dependence of $k_{\rm tr}$)

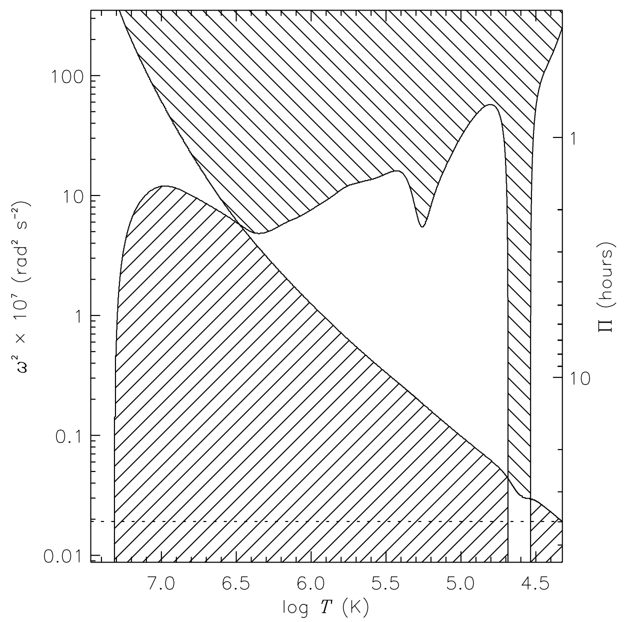

Figure 1. The propagation diagram for $l = 4$, $m = -1$ modes in the 7-$M_\odot$ stellar model, plotted in the $(\log T, \omega^2)$ plane to emphasize the outer regions of the model; the period Π corresponding to the frequency is shown on the right-hand ordinate. Hatched areas correspond to propagative regions where waves have g-mode (/) or p-mode (\) character, whilst evanescent regions are blank. The dotted horizontal line shows the position of the trapping cut-off frequency $\omega_{\rm t}$.

Table 1. The physical parameters of the stellar model considered throughout.

$M/M_\odot$	$R/R_\odot$	$L/L_\odot$	$T_{\rm eff}$/K	Z
7.00	3.16	1.76×10^3	21 000	0.02

and is relatively small at the surface, while N^2 is approximately zero in the convective core ($\log T \gtrsim 7.3$) and relatively large at the surface due to the steep stratification there. The prominent 'well' at $\log T \approx 4.6$ indicates the presence of a thin convective region ($N^2 < 0$) due to He II ionization, whilst the smaller well at $\log T \approx 5.3$ is due to the metal opacity bump responsible for κ-mechanism pulsation in early-type stars.

As mentioned in the previous section, the $k_r^2 = 0$ curves which separate propagative and evanescent regions correspond to the reflective boundaries required for wave trapping. Inspection of Fig. 1 shows that, for g modes with frequencies greater than the trapping cut-off ω_t, where $\omega_t^2 \approx 1.9 \times 10^{-9}\,\mathrm{rad^2\,s^{-2}}$ is shown in the figure as a horizontal dotted line, there exists an extensive trapping zone formed by a pair of reflective boundaries, one at the edge of the convective core and the other in the envelope at lower temperatures. In contrast, for modes with frequencies below ω_t, waves are propagative even at the surface of the star, and the outer reflective boundary required for the formation of a trapping zone does not exist.

Strictly speaking, the term 'mode' is not appropriate in such circumstances, since stationary waves will not be established by repeated complete reflection. However, all waves at frequencies below the cut-off are evanescent in the convective region at $\log T \approx 4.6$. This region, with a width of approximately 0.16 per cent of the stellar radius, behaves like a partially-reflecting barrier to waves incident from the interior; some fraction of the waves will leak through the barrier and thence propagate unhindered to the surface, where they are lost from the star, whilst the remaining reflected fraction will contribute to the establishment of 'somewhat-stationary' waves interior to the barrier. Within the adiabatic approximation, these waves must decay exponentially in amplitude with time to compensate for the the energy lost through leakage, but will still exhibit a discrete eigenfrequency spectrum. Shibahashi & Osaki (1976), when considering a similar situation for high-frequency g modes in evolved early-type stars, drew a useful analogy with virtual levels in the potential problem of quantum mechanics; therefore, it seems appropriate to refer to such partially-trapped waves as *virtual modes*. Whether virtual modes can actually be self-excited in a star depends on the balance between the input of vibrational energy from a suitable driving mechanism, and the loss of vibrational energy associated with the leakage; non-adiabatic calculations are required to answer such a questions.

The trapping cut-off frequency ω_t, which separates the leaking virtual modes from the fully-trapped 'traditional' modes, is given by the smaller root of the dispersion relation (13) at the stellar surface, namely

$$\omega_t^2 = k_{\mathrm{tr}}^2 c_{\mathrm{s}}^2|_{r=R}, \tag{19}$$

where R is the stellar radius. This expression demonstrates the pivotal role of the effective transverse wavenumber k_{tr}, discussed at the end of the preceding section, in determining the trapping condition at the surface. In the non-rotating context, k_{tr} can be eliminated from this expression through use of equation (14) to give

$$\omega_t^2 = \left.\frac{l(l+1)c_{\mathrm{s}}^2}{r^2}\right|_{r=R} \tag{20}$$

for $\Omega = 0$.

When the effects of rotation are included, the more general expression (12) for k_{tr} must be used in evaluating the sign of k_r^2 using the dispersion relation (13). However, propagation diagrams may be constructed and interpreted in exactly the same manner as the non-rotating case. Fig. 2 shows the propagation diagram for the 7-$M_\odot$ stellar model considered previously, but with rotation included at an angular frequency $\Omega = 8.04 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$, which is half of the critical rotation rate for the star, and corresponds to a period of 21.7 h. The effects of the rotation on the equilibrium stellar structure having been neglected. Calculation of the eigenvalue λ_{lm} in equation (12), for each frequency ordinate value in the $(\log T, \omega^2)$ plane, was accomplished using Townsend's (1997) implementation of the matrix eigenvalue method presented by Lee & Saio (1990). This method corresponds to the spectral expansion of Hough functions in a truncated series of associated Legendre polynomials of the same azimuthal order m; 100 expansion terms were used throughout the calculations, a value deemed to provide sufficient accuracy since a similar calculation with 200 terms produced no numerical change in the results. An azimuthal order $m = -1$ was adopted, so Fig. 2 should be taken as appropriate for modes with $(l, m) = (4, -1)$.

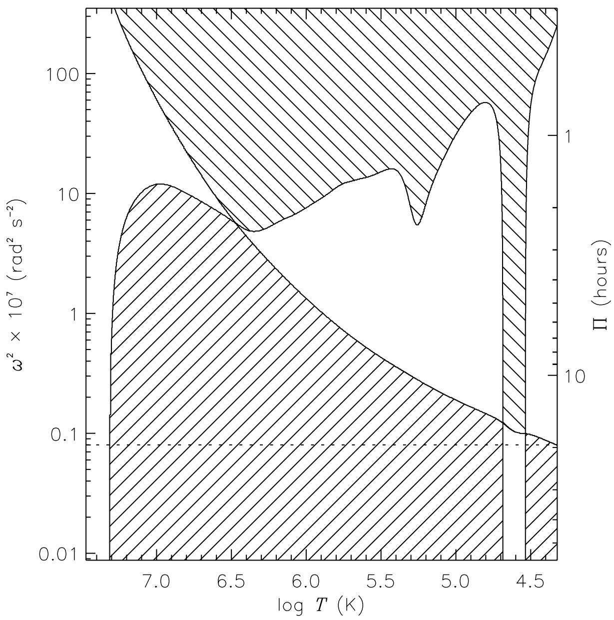

Figure 2. As for Fig. 1, but rotation has been introduced at an angular frequency $\Omega = 8.04 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$. The azimuthal order m is -1.

Inspection of this figure shows that the trapping cut-off is significantly larger ($\omega_t^2 \approx 8.0 \times 10^{-9}\,\mathrm{rad^2\,s^{-2}}$) than in the non-rotating case. This is a direct consequence of the influence of rotation on k_{tr}; at low frequencies where $\omega < 2\Omega$, $\nu > 1$, and k_{tr} can assume large values, as discussed in the preceding section. The appropriate expression for ω_t in rotating stars is given by

$$\omega_t^2 = \left.\frac{\lambda_{lm} c_s^2}{r^2}\right|_{r=R}, \tag{21}$$

although this should be regarded as formal, since it must be remembered that λ_{lm} is itself a function of ω through its dependence on the parameter ν. However, in the limit $\Omega^2 \gg c_{\mathrm{s}}^2/r^2$ (at the surface), this expression will have solutions corresponding to $\nu \gg 1$, and thus the asymptotic expressions (16,18) found previously may be used in the place of the general expression (19) for k_{tr}. Solving the resulting equations for ω_t then gives

$$\omega_t^2 = \begin{cases} 2\Omega(2l - 2|m| + 1)c_{\mathrm{s}}/r & (m > 0) \\ 2\Omega(2l - 2|m| - 1)c_{\mathrm{s}}/r & (-l < m \leqslant 0) \\ m^2 c_{\mathrm{s}}^2/r^2 & (m = -l) \end{cases}\Bigg|_{r=R} \tag{22}$$

for $\Omega^2 \gg c_s^2/r^2$. Applying the middle expression to the 7-$M_\odot$ model for $\Omega = 8.04\times10^{-5}$ rad s^{-1}, and $c_s/r = 9.76\times10^{-6}$ s^{-1} at the surface, leads to the asymptotic value $\omega_t^2 = 7.85\times10^{-9}$ rad^2 s^{-2} for $(l,m)=(4,-1)$ modes, which is in reasonably good agreement with the value $\omega_t^2 \approx 8.0\times10^{-9} rad^2\, s^{-2}$ shown in Fig. 2.

The above expressions, when compared with equation (20), demonstrate that the effect of rotation is to increase the trapping cut-off ω_t for all but the prograde sectoral modes; these latter modes will exhibit a smaller cut-off in rotating stars than in the non-rotating case, owing to their transformation into Kelvin waves discussed previously. This result is interesting in light of anecdotal observational evidence favouring prograde sectoral modes as the source of periodic line-profile variations in rapidly-rotating early-type stars. If such evidence can be substantiated at a quantitative level, as has been done by Howarth et al. (1998) for the rapidly-rotating pulsators HD 93521 and HD 64760, then it can be suggested that the bias towards prograde sectoral modes is due to the suppression of other types of mode, which will have a large values of ω_t at rapid rotation rates and therefore preferentially leak from the star without self-excitation.

5 EIGENFREQUENCIES

In addition to its influence on the trapping cut-off frequency ω_t, rotation modifies the eigenfrequencies and eigenfunctions of individual modes through its influence on the positions of trapping boundaries; this can be anticipated from the appearance of λ_{lm} in the pulsation equations (3–4). To evaluate the modified eigen-frequencies at a qualitative level, the asymptotic technique developed by Shibahashi (1979) and Tassoul (1980) may be adapted using the traditional approximation to given expressions appropriate for rotating stars. Using such an approach, Lee & Saio (1987) found that low-frequency g modes trapped between the boundary of the convective core ($r = r_c$) and the surface of a rotating early-type star have eigenfrequencies ω_n given by

$$\omega_n = \frac{2\sqrt{\lambda_{lm}}}{(n+\eta_e/2-1/6)}\int_{r_c}^{R}\frac{|N|}{r}\,\mathrm{d}r, \tag{23}$$

where η_e is the effective polytropic index at the surface, and n is the radial order of the mode. This expression is not strictly appropriate in the current context, owing to the fact that the outer reflecting boundary for trapped modes in Figs 1 and 2 occurs at $r < R$; furthermore, the presence of the convective region at $\log T \approx 4.6$ has been neglected. However, the form of the expression demonstrates that the frequencies of individual modes share the same λ_{lm}-dependence as the trapping cut-off ω_t in equation (21).

It can therefore be suggested that a g mode that is trapped in a non-rotating star will remain trapped once rotation is introduced, since the effect of rotation is to scale both sides of the trapping condition $\omega_n > \omega_t$ by an equal amount. Disregarding the possibility of avoided crossings (Aizenman, Smeyers & Weigert 1977; Lee & Saio 1989), a more general hypothesis may be put forward that, *ceteris paribus*, the set of radial orders $\{n\}$ of the g modes which are trapped in a non-rotating star will remain invariant under the influence of rotation, with a similar result applying to the virtual modes. The hypothesis can be supported with an analogy drawn to atomic energy levels under the influence of a magnetic field; even though the levels are distorted by the action of the Lorentz force (which, like the Coriolis force, can be expressed as a velocity cross-product), the set of discrete states which are bound is invariant under the action of the field. However, numerical calculations should be employed to test this hypothesis rigorously.

6 DISCUSSION

An important caveat regarding *quantitative* interpretation of the results presented previously is that atmospheric layers above the photosphere at $\tau = 2/3$ have been disregarded. This is justifiable if k_{tr} is constant throughout these layers; however, such a situation is unlikely to be realized, since even in the case of isothermal trans-photospheric regions where the sound speed c_s is constant, $k_{tr} \sim 1/r$ because of the spherical geometry. As a consequence, waves which are formally propagative at $r = R$ may leak out to some radius $r > R$ and thence be reflected back towards the interior, leading to complete wave trapping at frequencies *below* the cut-off ω_t.

To address properly this issue of trans-photospheric reflection, it is necessary to relocate the nominal outer boundary of the star to a radius at which it is guaranteed that no reflected, inward-propagating waves will occur. In the context of the linear and adiabatic approximations adopted herein, this guarantee can only be made of the outer boundary is located at infinity. However, once non-adiabatic effects are considered, it is possible that strong radiative or non-linear dissipation above the photosphere can lead to the effective absorption of all outward-propagating waves, with no reflection and subsequent trapping. Such a situation is analogous to the core-absorption of inwardly-propagating envelope p modes found by Osaki (1977), and can be treated by placing the outer boundary at the base of the dissipative region (which may be close to the photosphere).

Similar arguments concerning the absence of trans-photospheric reflection can be made for systems with stellar winds. Owocki & Rybicki (1986) found that, for a line-absorption driven wind, any wave-like disturbance which reaches the sonic point r_s can never propagate back to smaller radii, due to the non-linear interaction between the wave and underlying mean flow. In this case, the outer boundary can be located at r_s; however, quantitative treatments are problematical, since the pulsation equations must be revised to take the sub-sonic wind regions into account.

In spite of these difficulties, the results of this work are valid on a phenomenological level, and may be of particular relevance to the 53 Per (Smith & Karp 1976) and slowly-pulsating B (SPB; Waelkens 1991) classes of variable stars, which are unstable to low-frequency g-mode pulsation owing to the metal opacity bump at $\log T \approx 5.3$ (Dziembowski & Pamyatnykh 1993; Dziembowski, Moskalik & Pamyatnykh 1993); the observed periods of these stars are typically 1-3 d, which is the same order of magnitude as the trapping cut-off depicted in figs. 1 and 2. As indicated previously, the global self-excitation of a given pulsation mode in one of these stars (or, indeed, any other type of star) depends on the competitive interplay between excitation and damping mechanisms, which, respectively, pump energy into and remove energy from the pulsation at each point within the star; self-excitation will only occur if the net contributions from the former outweighs the net deductions from the latter. For the purposes of the following discussion, both of these generic energy-transfer processes may be classified as either

(a) non-adiabatic, where the transfer arises through perturbations to the specific entropy, corresponding to the operation of a

Carnot heat engine which converts between thermal and mechanical (wave) energy within a given region, or

(b) advective, where the transfer arises through the non-zero divergence of the wave flux, corresponding to a net flow of mechanical energy through the boundaries of the region.

The opacity mechanism operative in 53 Per and SPB stars is thus a non-adiabatic excitation mechanism (a), whilst the energy loss associated with wave leakage at frequencies below the trapping cut-off ω_t may be identified as an advective damping mechanism (b). In the stability calculations of Dziembowski et al. (1993), who use the approach described by Dziembowski (1977), the assumption is made that the Lagrangian pressure perturbation δp tends to a limiting value at the surface; this corresponds to the *ab initio* restriction that all waves are evanescent at the surface, and thus completely trapped within the star. Hence, any contributions to advective damping arising from wave leakage are neglected, which *might* lead to incorrect results for the overstability of modes at frequencies below ω_t.

However, it must be stressed that the last point is somewhat formal, if modes are stabilized by non-adiabatic damping well before the frequency is low enough for leakage to occur; whilst advective damping might enhance the stability of the virtual modes, it will have little influence in determining which (trapped) modes are unstable in a star. In the case of the 53 Per and SPB stars, such a situation may arise due to the dominance of the opacity mechanism by radiative damping at lower frequencies. The latter is large for high-order (large-n) g modes, whose eigenfunctions exhibit many radial nodes in the stellar envelope; the sub-adiabatic temperature gradient in the radiative parts of the envelope will lead to significant thermal diffusion between neighbouring oscillating elements, which tends to suppress the pulsation (see, for instance, equation 26.13 in Unno et al. 1989 plus their accompanying text). These issues will be examined further in the next paper in this series.

In a contrasting situation, where non-adiabatic damping is less important at low frequencies, the overstability of virtual modes will be determined by the relative strengths of non-adiabatic excitation and advective damping. If the latter is dominant, then g modes will exhibit an upper limit in their variability periods which corresponds to the trapping cut-off; in the more rapidly-rotating stars (see, e.g., Aerts et al. 1999), this upper limit will depend, amongst other things, on m and the degree of rotation. In contrast, if non-adiabatic excitation dominates, then no upper period limit will be observed, since virtual modes will be excited in addition to trapped modes. Estimates of the strength of advective damping can be obtained using, for instance, the asymptotic approach presented by Shibahashi (1979). However, as with the local analysis used in Section 3, such an approach is only valid when the characteristic variation scale of eigenfunctions is much smaller than that of the underlying star. This restriction means that Shibahashi's approach may lead to poor results for those virtual modes with frequencies close to ω_t; therefore, an examination of the importance of leakage-originated advective damping is deferred to the following paper, where the pulsation equations are solved globally using a numerical approach which does not suffer from the restriction discussed.

Whilst a proper treatment of trapping, even in cases without the trans-photospheric reflection described above, adds a certain level of complexity to theoretical studies, it does open the way for asteroseismological studies of the near-surface regions of early-type stars. For instance, if an upper period limit is observed as described, the inferred value of ω_t may be used, in tandem with equations (20–22), to calculate a value for the acoustic time-scale

$$\tau_{\mathrm{acc}} = r/c_{\mathrm{s}} \tag{24}$$

in the region where the onset of wave leakage occurs. Since, for an ideal gas, the adiabatic sound speed c_s is a function of temperature T alone, this time-scale then gives an independent estimate of the temperature in the outer layers of the star. Conversely, observations of variability attributable to virtual modes can confirm the existence of sub-surface convective regions arising in ionization zones, predicted by evolutionary models of early-type stars. The degree of wave leakage associated with a virtual mode depends on the thickness of these regions (which form the partially-reflective barrier necessary for the existence of virtual modes); therefore, it might be possible to obtain estimates of the thickness through measurements of the leakage rate.

7 CONCLUSIONS

The prime conclusion to be drawn from the work presented herein is that the complete trapping of low-frequency g modes beneath the surface of early-type stars is not guaranteed. This is especially the case in rotating stars, where the trapping cut-off frequency ω_t can be significantly increased by the action of the Coriolis force for all but the prograde sectoral modes. The fact that the latter are more effectively trapped in rapidly-rotating stars than other types of modes may explain anecdotal observational evidence which points to their favoured excitation. As a consequence of the dependence of g-mode eigenfrequencies on the rotation rate, the hypothesis has been put forward that the set of radial orders $\{n\}$ of trapped g modes is invariant under the influence of rotation.

Stability analyses which contain the *ab initio* assumption of complete wave reflection at the stellar surface might be in error at frequencies below the cut-off ω_t. More rigorous calculations can include the possibility of wave leakage, by adopting a more physically-realistic outer mechanical boundary condition. Such calculations will reveal to what extent advective damping associated with leaking virtual modes might suppress the self-excitation of these modes. These points may be of especial relevance to the 53 Per and SPB classes of variable stars.

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