The Pulsation-Rotation Interaction: Greatest Hits and the B-side

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A

How does rotation affect pulsation?

THE NONRADIAL OSCILLATIONS OF GASEOUS STARS AND THE PROBLEM OF BETA CANIS MAJORIS*

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$$\omega = \omega_0 + m\Omega(1 - C_{n,\ell})$$

Inertial Frequency

Unperturbed Frequency

Doppler Shift

Coriolis Perturbation

Third-Order Perturbation Theory

$$W = y_k y_q Y_k^* Y_q \left(P_2 w_1 + (1-P_2)\frac{2}{3} r \frac{d\Omega^2}{dr}\right) - \frac{1}{3} r \frac{d\Omega^2}{dr}\left(y_k z_q Y_k^* \boldsymbol{\nabla}_H Y_q + y_q z_k Y_q \boldsymbol{\nabla}_H Y_k^*\right).\boldsymbol{\nabla}_H P_2 \quad \text{(B9)}$$

with P_2 being the second order Legendre polynomial and

$$w_1 = \frac{d}{dr}\left(\frac{1}{\rho}\right)\frac{dp_{22}}{dr} + \frac{1}{\rho}\frac{d}{dr}\left(\frac{\rho_{22}}{\rho}\right)\frac{dp}{dr} \quad \text{(B10)}$$

The detailed expression of $\mathcal{D}_{kq}$ involves derivatives of density which tend to magnify surface effects and can introduce inaccuracies in the computation of the frequencies when the atmosphere is poorly described. This is particularly true for high overtones which are more concentrated toward the outer layers. Density derivatives are therefore eliminated by means of integrations by parts and with the use of the following relations:

$$\frac{d\Gamma_1 p\lambda}{dr} = -\rho\, g\, q; \qquad q = c - y_t h_1$$
$$c = y\,(C\hat{\sigma}^2 + 4 - U) - \Lambda z - w \quad \text{(B11)}$$

As a result of integrations by parts, one finally obtains:

$$\mathcal{D}_{kq} = \omega_0 \left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \left(\delta_{l_k,l_q}\, \mathcal{D}_1 + \mathcal{Q}_{kq2}\left(\mathcal{D}_2 + 2m\frac{\overline{\Omega}}{\omega_0}\mathcal{D}_3\right)\right) \quad \text{(B12)}$$

with

$$\mathcal{D}_1 = \frac{\omega_0}{2J}\int dr \rho r^4\, y_k y_q b_2$$
$$\mathcal{D}_2 = \frac{\omega_0}{2J}\int dr \rho r^2\, \frac{1}{2}(D_{qk} + D_{kq}) \quad \text{(B13)}$$
$$\mathcal{D}_3 = \frac{\omega_0}{2J}\int dr \rho r^2\, (D_{3kq} + D_{3qk}) d_1$$

where

$$\begin{aligned} D_{3kq} &= \lambda_k y_{t_q} + \frac{1}{2} y_k y_{t_q}(U-4) \\ &+ \frac{1}{2} C\sigma_0^2\, \left(y_k y_q + (\bar{\Lambda}-3) z_k z_q\right)\, \frac{d\ln\Omega}{d\ln r} \\ &- \frac{1}{\Lambda_q}(\bar{\Lambda}-3)(z_q + y_q) C\sigma_0^2 (y_k - 2z_k) \qquad \text{(B14)} \\ &+ \frac{1}{\Lambda_q}(\bar{\Lambda}-3)(z_q+y_q)\frac{A}{V_g}\lambda_k + (\bar{\Lambda}-3) z_k y_{2q} \\ D_{qk} &= -(d_1 F_1 + d_2 F_2 + r^2 b_2 F_3 + r^2 b_3\, y_k s_q) \end{aligned}$$

and

$$\begin{aligned} &F_1 = y_k y_q a_1 + \lambda_k a_2 + a_3 y_k + a_4 z_k + a_5 z_k z_q + a_6 y_{tq} \\ &F_2 = y_k y_q (U - C\sigma_0^2) + y_q[2w_k + (4-2U)s_k - q_k] \\ &+ z_q[\Lambda_q(v_k - y_k) + (\Lambda_k - 6) y_{tk}] - C\sigma_0^2(\bar{\Lambda}-3)\, z_k z_q \\ &F_3 = C\sigma_0^2(y_q y_k + z_q z_k(\bar{\Lambda}-3)) + y_q s_k (U+6) \qquad \text{(B15)} \\ &+ (\Lambda_k - \Lambda_q + 6) z_k y_q + (6 - 2\bar{\Lambda}) z_k v_q \\ &- 2y_q w_k - s_k \lambda_q \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho}\right)_p \end{aligned}$$

The a_j quantities in F_1 above are defined as:

$$\begin{aligned} a_1 &= 6 - U\left(U + 2\,\frac{d\ln\rho}{d\ln r}\right) + C\sigma_0^2\, U \\ a_2 &= -2U y_q \\ a_3 &= -2U w_q + 2\Lambda_q v_q + (U-4)(c_q + 2\Lambda_q z_q) + s_q \psi \qquad \text{(B16)} \\ a_4 &= 2\Lambda_k w_q + \Lambda_k (4-U) s_q + 2y_{tq}(\bar{\Lambda}-3)(4-U) \\ &- (\Lambda_k - \Lambda_q + 6)\, y_q C\sigma_0^2 + 2(\bar{\Lambda}-3) U \delta_{q,0} y_q \\ a_5 &= C\sigma_0^2\, (U-6)(\bar{\Lambda}-3) \end{aligned}$$

$$a_6 = 2\delta_{lq,0}(1 - \delta_{lk,0})\,(\bar{\Lambda}-3)\left(y_k - 2z_k - \frac{A}{V_g}\frac{\lambda_k}{C\sigma_0^2}\right)$$

For shortness, we have defined:

$$\begin{aligned} \psi &= 6 + U(U-3) + (4-U)(1-U) \\ \bar{\Lambda} &= \frac{\Lambda_k + \Lambda_q}{2} \\ s &= y - y_t + v; \qquad \lambda = V_g(y - y_t + v) \\ d_1 &\equiv r^2 u_2;\ \ d_2 \equiv r\frac{dr^2 u_2}{dr};\ \ b_2 = \tfrac{1}{3} r \frac{d\eta_2}{dr};\ \ b_3 = \tfrac{1}{3} r^2 \frac{d^2\eta_2}{dr^2} \qquad \text{(B17)} \end{aligned}$$

We recall that d_1, d_2 arise from the non-spherically symmetric distortion of the equilibrium model and must be obtained by numerical integration of the system Eq. (17).

Integration over $\theta,\ \varphi$ imposes selection rules upon the degrees of the near-degenerate modes which can couple. The azimuthal order m must be the same and one has

$$\mathcal{Q}_{kq2} = \int \sin\theta\, d\theta\, d\varphi\, Y_k^* Y_q P_2$$
$$= \delta_{l_k,l_q}\mathcal{Q}_{kk2} + \frac{3}{2}\delta_{l_k,l_q+2}\,\beta_k\beta_{q+1} + \frac{3}{2}\delta_{l_k,l_q-2}\,\beta_{k+1}\beta_q \quad \text{(B18)}$$

$$\mathcal{Q}_{kk2} = \frac{3}{2}\,(\beta_{k+1}^2 + \beta_k^2) - \frac{1}{2} = \frac{\Lambda_k - 3m^2}{4\Lambda_k - 3}$$

The diagonal coefficient $\omega^D \equiv \mathcal{D}_{kk}$ is:

$$\omega^D = \left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \left(\, \omega_0 J_2^D + m\bar{\Omega} J_3^D + m\bar{\Omega} J_2^D\, (C_L - 1 - J_1)\,\right) \quad \text{(B19)}$$

with

$$\begin{aligned} J_2^D &= J_{2D,1} - Q_{kk2}\, J_{2D,2} \\ J_3^D &= -Q_{kk2}\, J_{3D,1} \qquad \text{(B20)} \end{aligned}$$

and

$$\begin{aligned} J_{2D,1} &= \frac{1}{2I}\int dr\, \rho r^4\, b_2\, y^2 \\ J_{2D,2} &= \frac{1}{2I}\int dr\, \rho r^2 \left(d_1 F_1 + d_2 F_2 + r^2 b_2 F_3 + r^2 b_3 F_4\right) \\ J_{3D,1} &= \frac{1}{I}\int dr\, \rho r^2\, D_{3kk} \qquad \text{(B21)} \end{aligned}$$

"We feel perturbation theory calculations are still useful... Undoubtedly, the use of [2-D hydrocodes] will ultimately be unavoidable, but then it will be very helpful to have a code based on the perturbational approach for comparisons at moderate equatorial velocities where both are valid."

Soufi et al. (1998)

Non-Perturbative Techniques

- Solve PDEs directly

  Savonije et al. (1995); Clement (1998)

- Expand in Y_l^m expand, solve coupled ODEs

  Durney & Skumanich (1968); Lee & Baraffe (1995); Reese et al. (2006); Ouazzani et al. (2012)

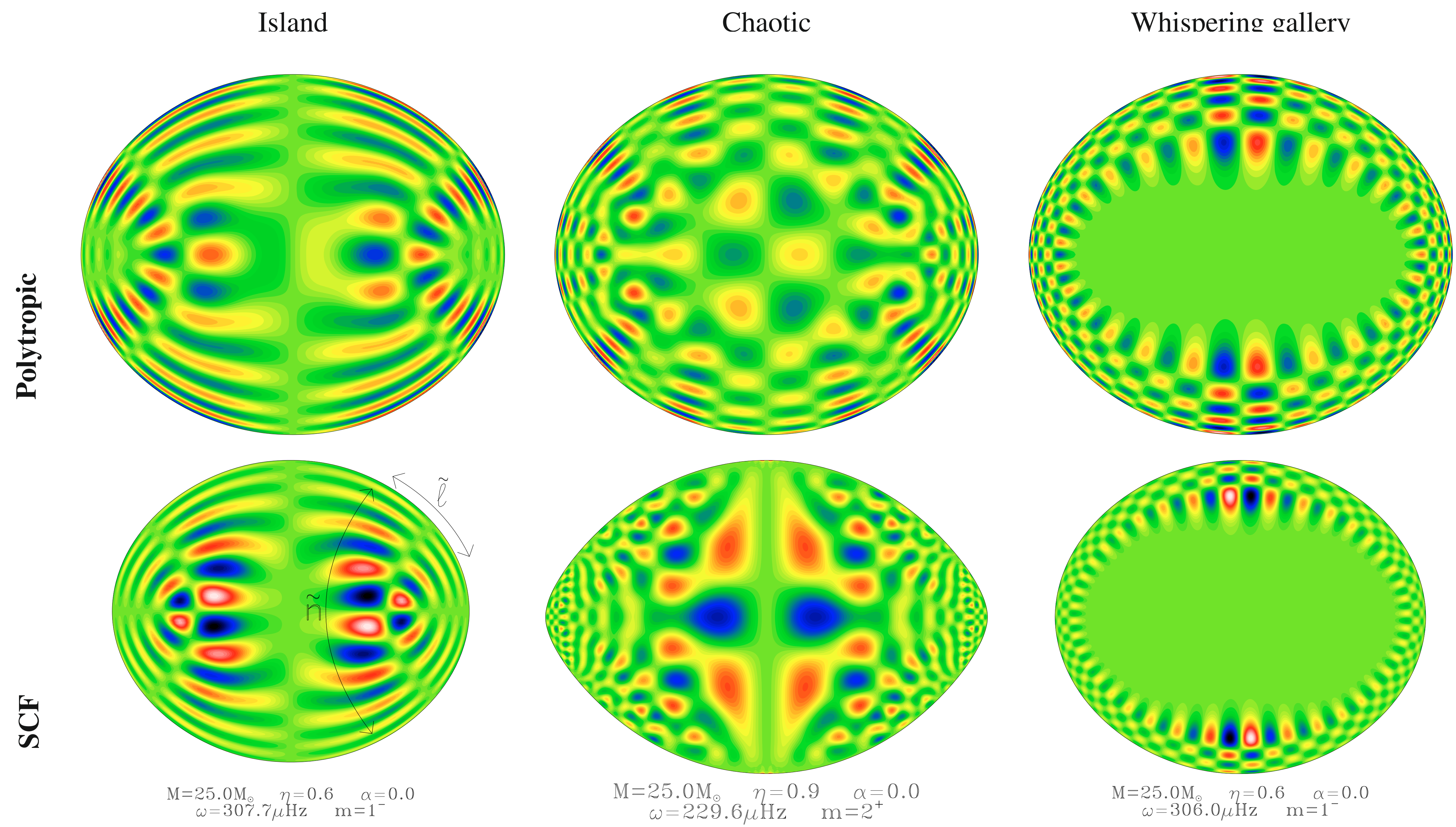
- Ray tracing (acoustic waves)

  Lignières & Georgeot (2008)

- Traditional approximation (gravity waves)

  Berthomieu et al. (1978); Bildsten et al. (1996); Lee & Saio (1997); Townsend (2003); Dziembowski et al. (2007); Mathis et al. (2008)

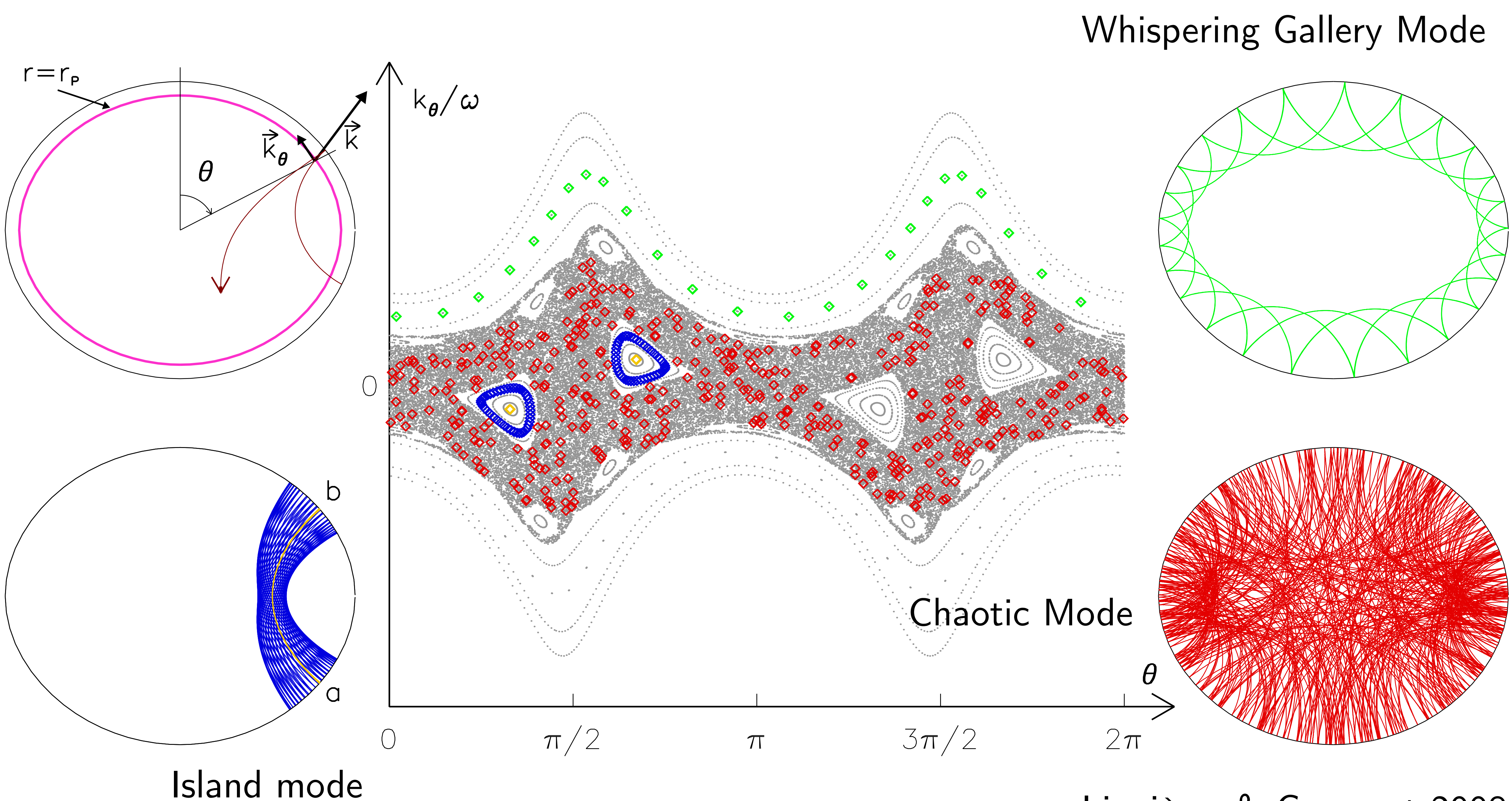
p-Modes in Rapidly Rotating Stars



Small ℓ - $|m|$ Intermediate ℓ - $|m|$ Large ℓ - $|m|$

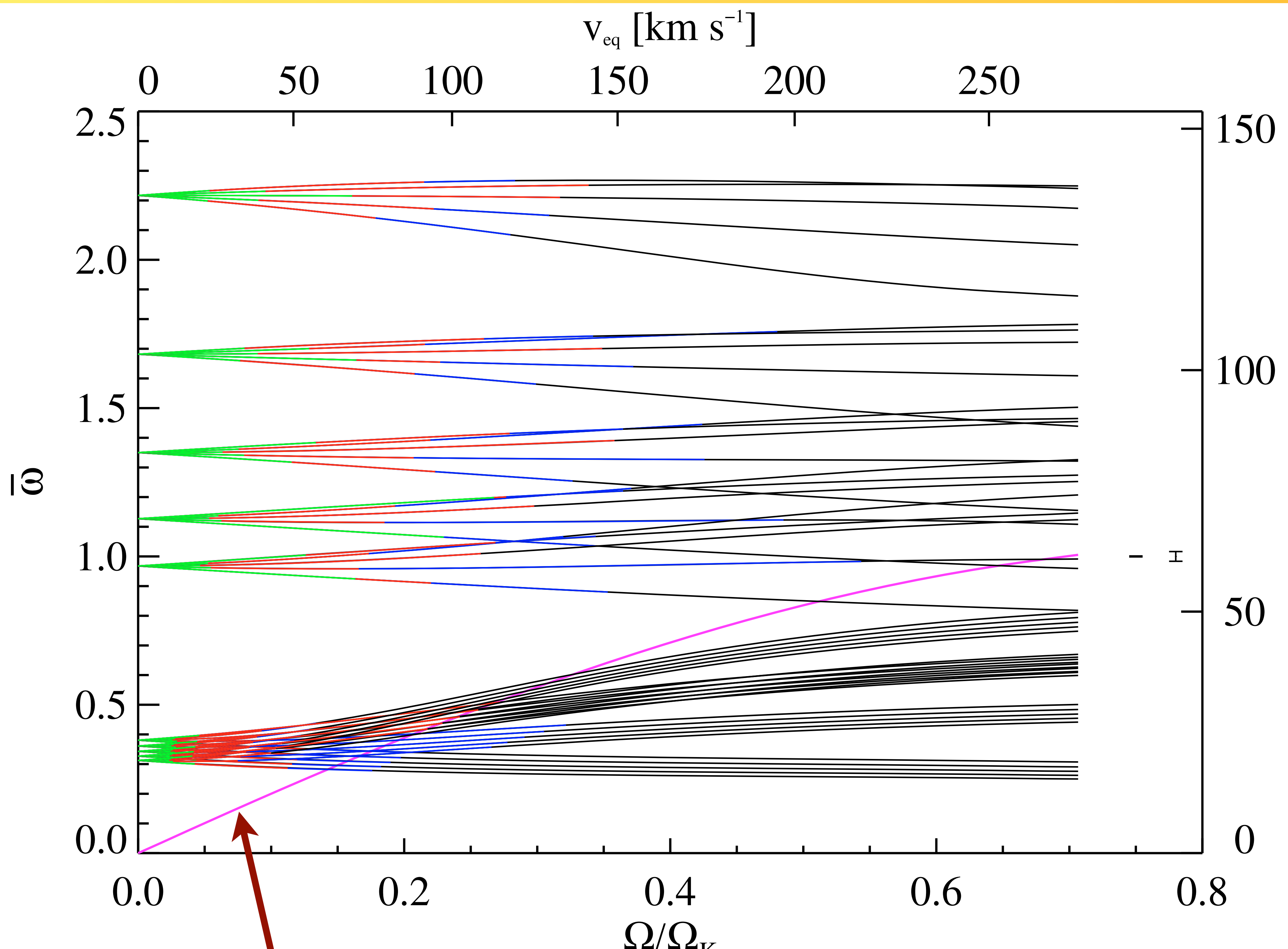
Reese et al. (2009)

Ray Tracing, Poincaré Section



Lignières & Georgeot 2008

Frequency Reorganization



Ballot et al. (2010)

$\omega = 2\Omega$

Frequency Reorganization: p-Modes

$$\omega \approx \tilde{n}\tilde{\Delta}_n + \tilde{\ell}\tilde{\Delta}_\ell + m^2\tilde{\Delta}_{\tilde{m}} - m\Omega_{\mathrm{fit}} + \tilde{\alpha}$$

Reese et al. (2009)

Even Parity

$$\tilde{n} = 2n$$

$$\tilde{\ell} = \frac{\ell - |m|}{2}$$

Odd Parity

$$\tilde{n} = 2n + 1$$

$$\tilde{\ell} = \frac{\ell - |m| - 1}{2}$$

The Spin Parameter

$$\nu = 2\frac{\Omega}{\omega_c}$$

...a measure of how much the star turns during one oscillation period

Inertial regime: $\nu > 1$

The Traditional Approximation

- Neglect the Coriolis force arising from the horizontal component of the rotation vector
- Valid when $\omega_c{}^2$, $\Omega^2 \ll N^2$ (g-modes)
- Pulsation equations identical to non-rotating case (with Cowling approx.), except:

$$\ell(\ell+1) \longrightarrow \lambda_{\ell,m}(\nu)$$

$$Y_\ell^m(\theta,\varphi) \longrightarrow \Theta_{\ell,m}(\theta;\nu)\exp(\mathrm{i}m\varphi)$$

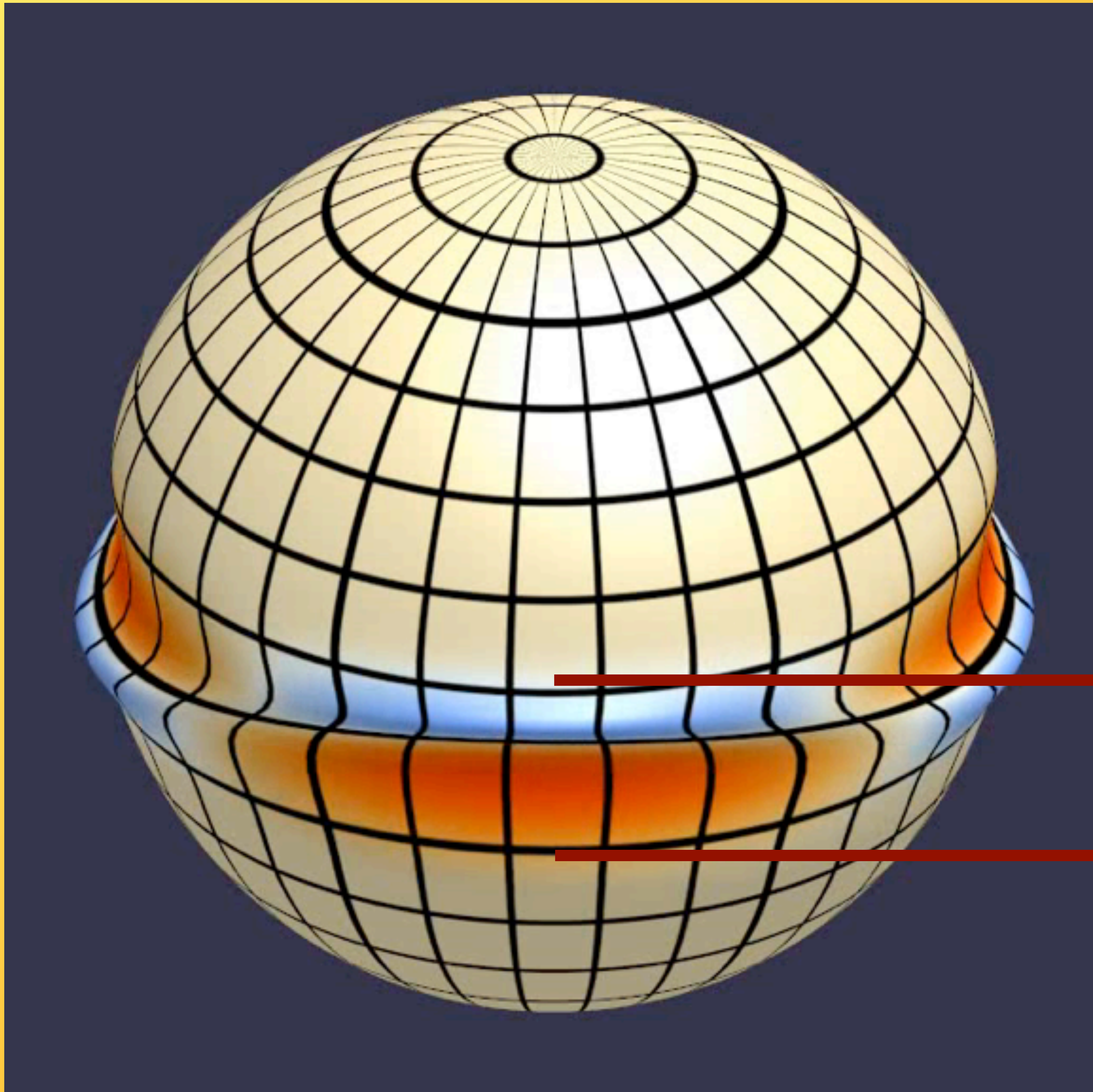
V. *On the Application of Harmonic Analysis to the Dynamical Theory of the Tides.—Part II. On the General Integration of* LAPLACE'S *Dynamical Equations.*

By S. S. HOUGH, *M.A., Fellow of St. John's College and Isaac Newton Student in the University of Cambridge.*

Communicated by Professor G. H. DARWIN, *F.R.S.*

Received October 27,—Read December 9, 1897.

The Equatorial Waveguide



Matsuno (1966)

$\cos(\theta) = \pm\nu$

Frequency Reorganization: g-Modes

$$\omega_c \approx \sqrt{(2\ell_\mu - 1)\Omega}\,\frac{W}{n}$$

Townsend (2003)

Prograde

$$\ell_\mu = \ell - |m|$$

Retrograde

$$\ell_\mu = \ell - |m| + 2$$

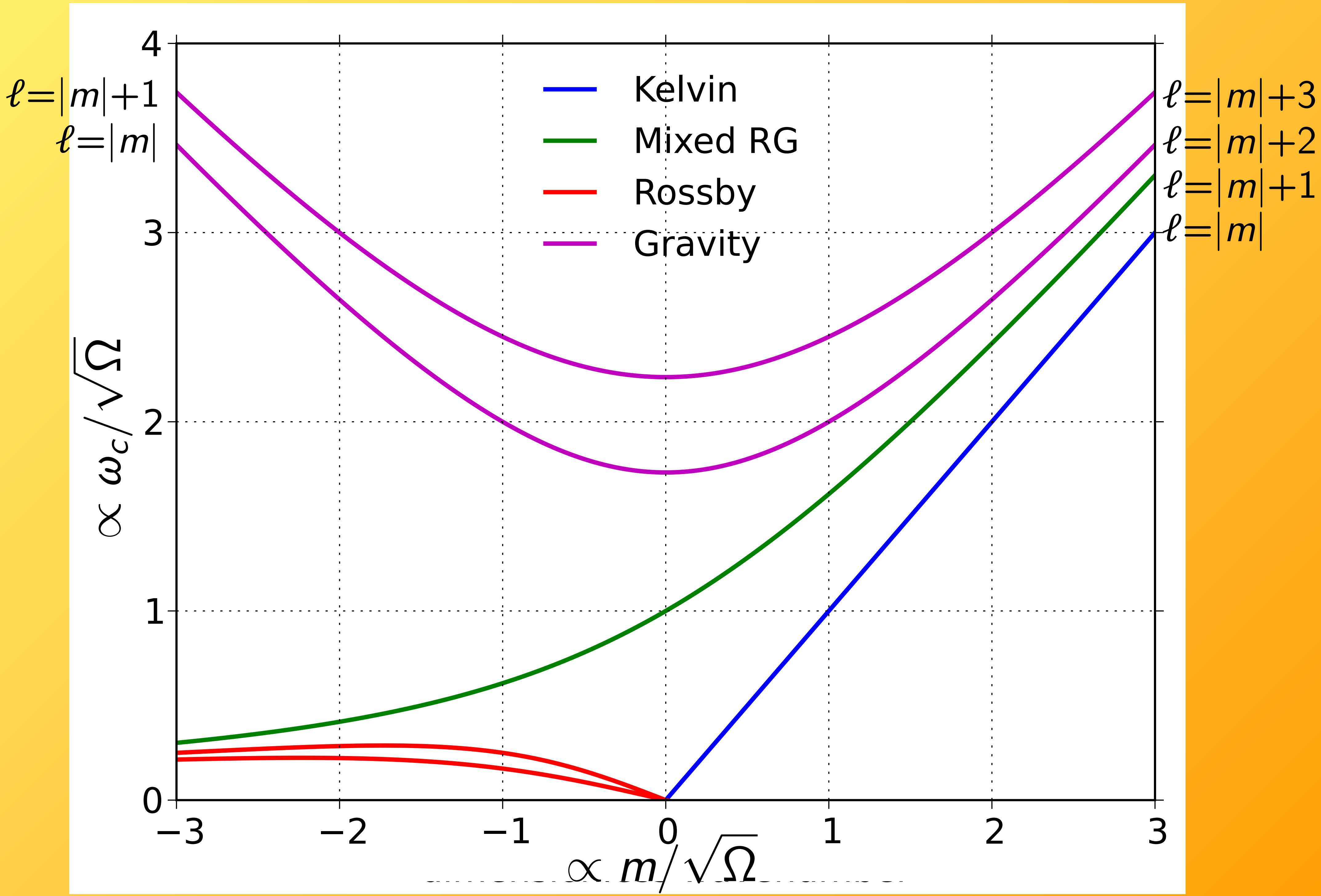
Note: NOT for prograde sectoral modes

Frequency Reorganization: Equatorial Kelvin Modes

$$\omega_c \approx m \frac{W_\mathrm{K}}{n}$$

- Become prograde sectoral g-modes in non-rotating limit
- Geostrophic: θ Coriolis force balances θ pressure gradients
- Independent of rotation rate
- Azimuthally dispersion-free (α Oph? KIC 8054146?)

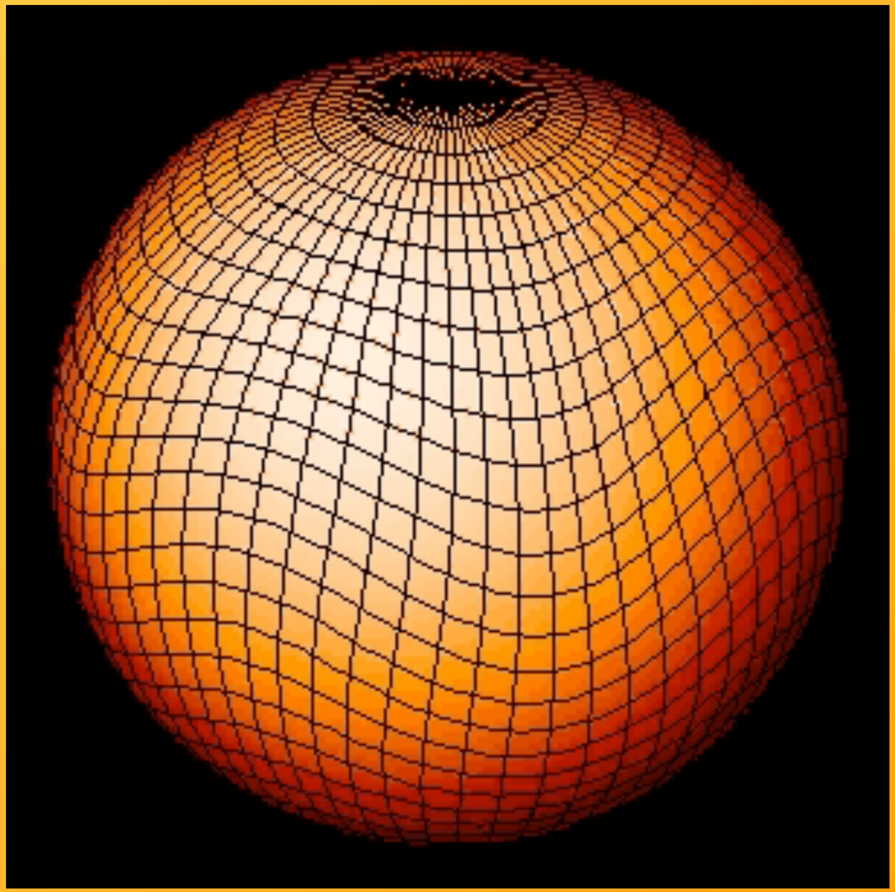
Azimuthal Dispersion Diagram



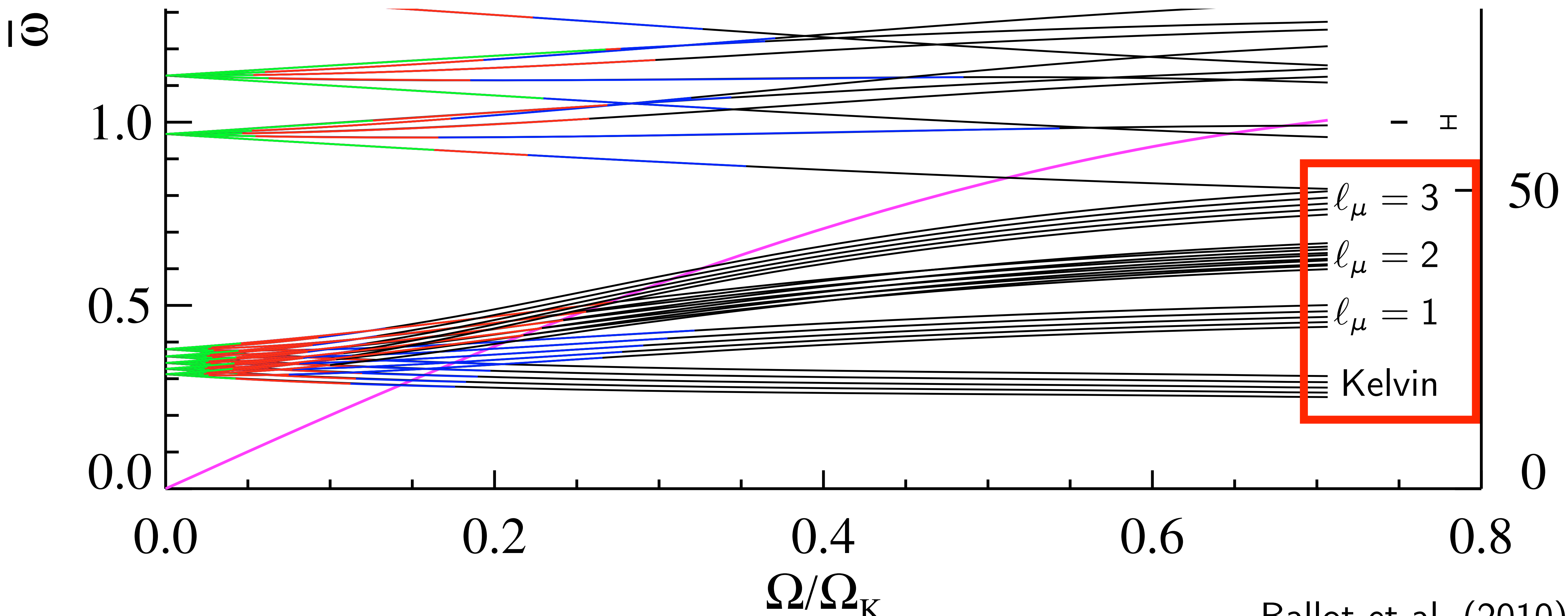
Slow → Rapid ← Slow

r-Modes

- Rotating versions of toroidal modes
- Propagate by conservation of vorticity
- Almost incompressible
  - No temperature perturbations
  - Difficult to excite by heat eng.

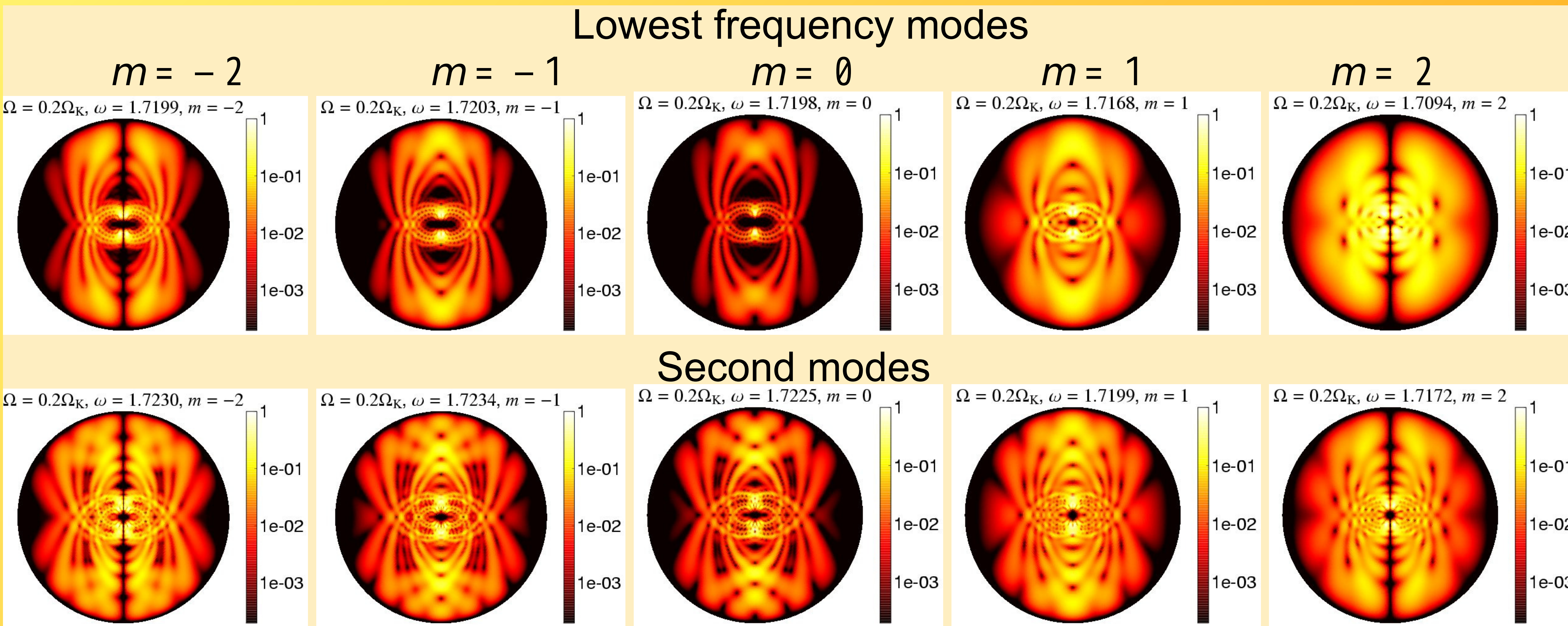


Frequency Reorganization Revisited



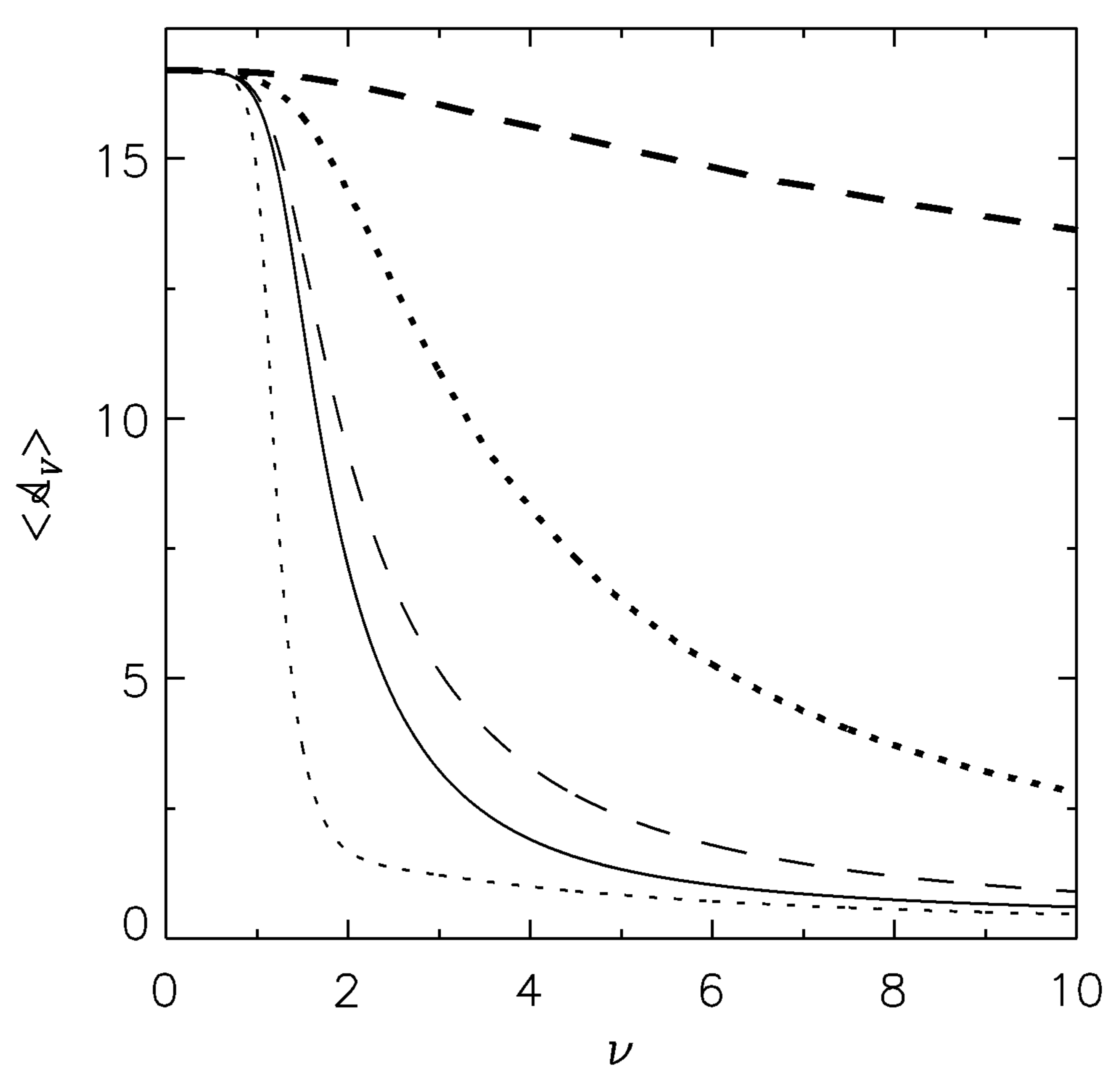
Ballot et al. (2010)

Rosette Modes: A Cautionary Tale



Takata & Saio (Poster #66)

Mode Visibilities



Townsend (2003)

- With rotation:
  - mode visibility generally decreases
  - often, modes are more visible from the poles
  - amplitude ratios become dependent on *m* and *i*
- More recent studies:
  - Daszyńska-Daszkiewicz et al. (2007)
  - Reese et al. (2012)

Differential Rotation: Critical Layers

Doppler shift: $\omega_c(r) = \omega - m\Omega(r)$

Dispersion relation: $k_r^2 \sim k_h^2 \dfrac{N^2}{\omega_c^2}$

$$m\Omega(r) \to \omega \qquad \omega_c(r) \to 0 \qquad k_r \to \infty$$

...the wave will be strong damped (absorbed) at the **critical layer**

(see poster #39)

B

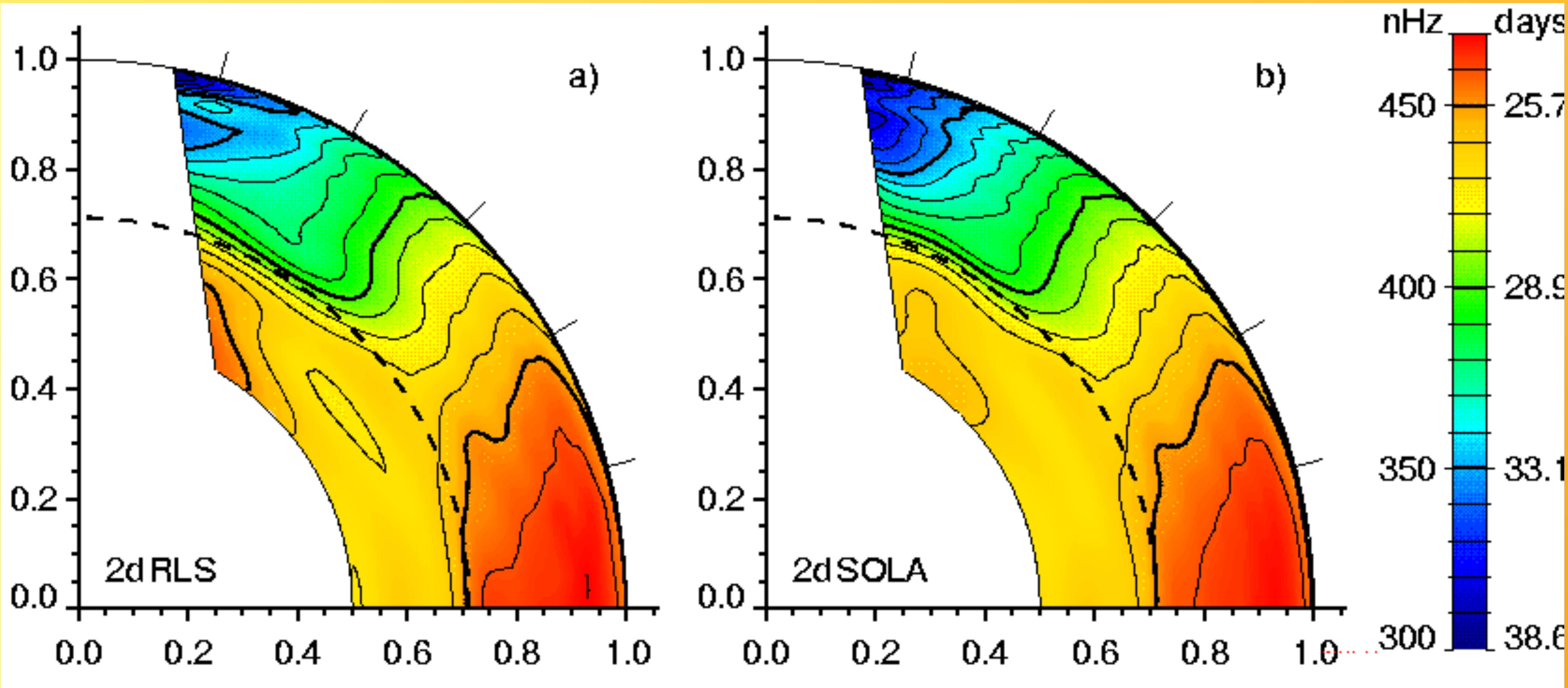
How does pulsation affect rotation?

Angular Momentum Transport by Pulsations

"...prograde modes can carry angular momentum from wave excitation regions to wave dissipation regions, and retrograde modes can do the contrary."

Ando (1983)

Uniform Rotation of the Sun's Interior



...evidence for J extraction by stochastic g modes?

(Talon et al. 2002)

Reynolds Decomposition of Azimuthal Momentum Equation

$$\frac{\partial}{\partial t}\langle \varpi \overline{\rho}\, \overline{v}_\phi \rangle = -\frac{1}{4\pi r^2}\frac{\partial}{\partial r} L_J - \frac{\partial}{\partial t}\langle \varpi \overline{\rho' v'_\phi} \rangle - \left\langle \overline{\rho' \frac{\partial \Phi'}{\partial \phi}} \right\rangle$$

Change in shell J	Divergence of J Luminosity	Change in wave J	Gravitational Torque

Angular Momentum Luminosity

(angular momentum passing per unit time through spherical shell)

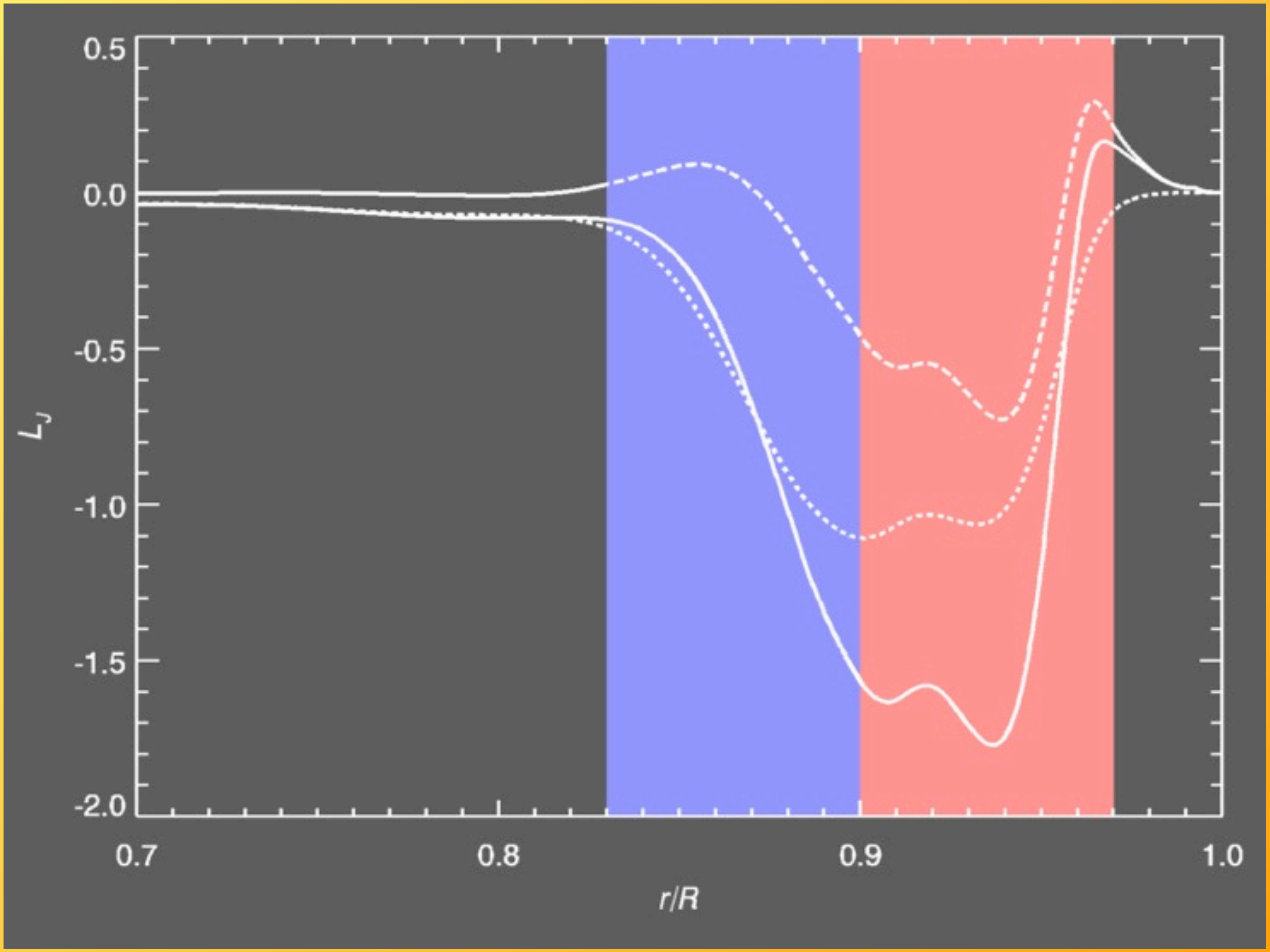
$$L_J = 4\pi r^2 \langle \varpi (\bar{\rho}\,\overline{v'_r v'_\phi} + \bar{v}_\phi\,\overline{\rho' v'_r} + \overline{\rho' v'_r v'_\phi}) \rangle$$

Reynolds Stress Flux

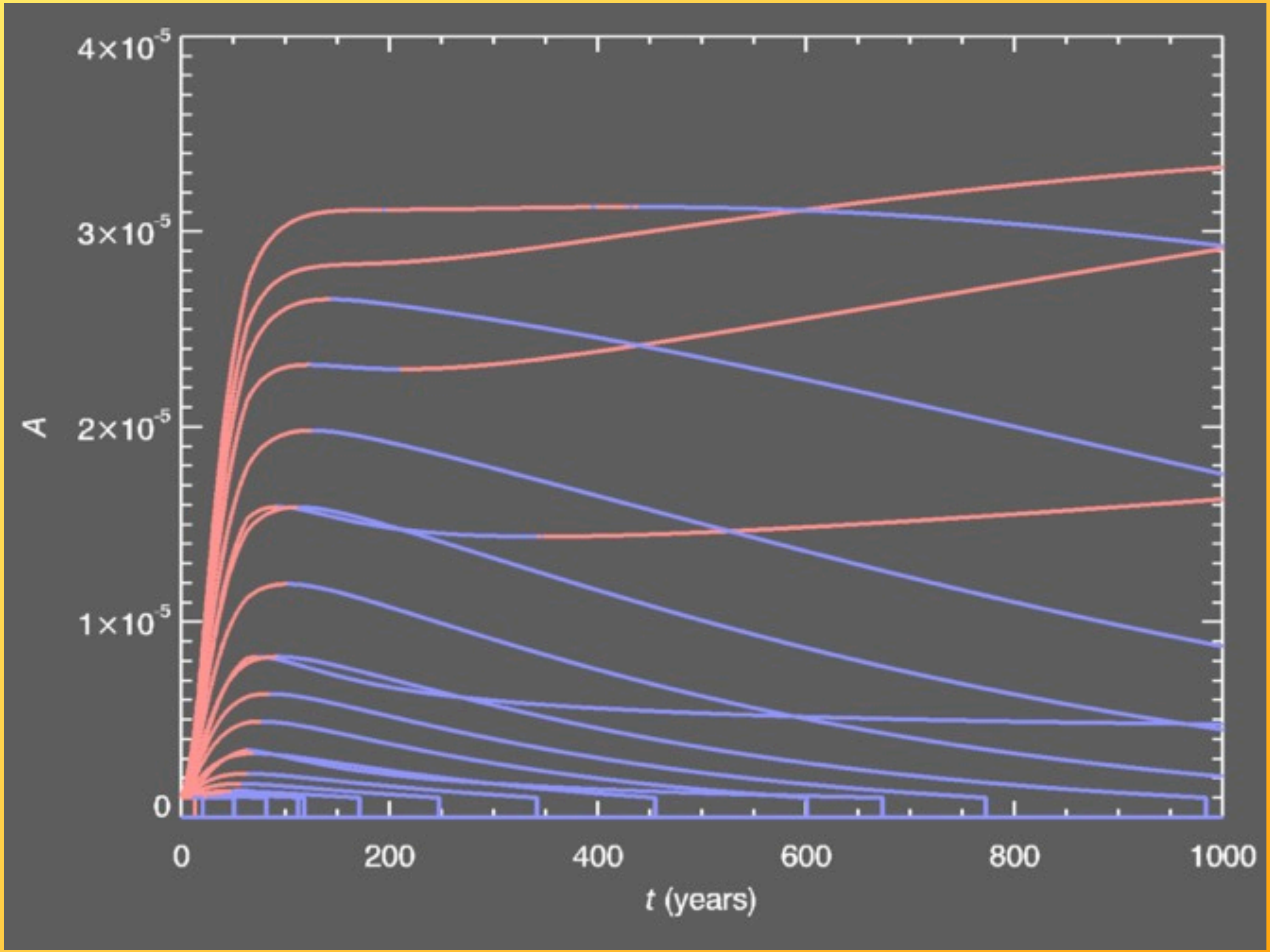
Eddy Mass Flux

Third Order Stuff

Example: L_J in a Massive MS Star



Amplitude Limitation via Interaction with Rotation?



Unstable → Stable

Just this year...

- Rogers et al. (2013): massive stars
- Alvan & Mathis (2013): critical layers (also, Poster #39)
- Mathis et al. (2013): interactions w/ other transport mechanisms
- Charbonnel et al. (2013): PMS stars

Summary

- The pulsation-rotation interaction is *tricky!*
- Nevertheless, much progress in past decade:
  - 2-D numerical modeling becoming widespread
  - Complementary tools are also emerging
  - Interest in J transport is taking off
- Crunch time: theory vs. observations