1 Introduction

The official course title of Astronomy 702 is 'Basic Astrophysics II' — but the real title should be 'Dynamics'. The topic of dynamics[1] concerns the time evolution of physical properties and processes. So, most of the equations we'll be encountering over the 14 weeks of the course involve a time derivative of one form or another.

2 A Hierarchy of Models

Many branches of physics and astrophysics focus on phenomena occurring at a certain scale. Here, I use 'scale' in two senses — on the one hand, the physical size of the system under consideration, and on the other, the number of interacting entities (particles, planets, etc.) comprosing the system. So,

3 Individual Particles

3.1 The Equation of Motion

The fundamental equation governing the dynamics of all particles in the classical (non-quantum) limit is Newton's Second Law, which relates the acceleration of an individual particle to the external forces acting upon it. Although we usually learn this as the simple

$$\mathbf{F} = m\mathbf{a},$$

the equation is in fact a differential equation involving time derivatives (remember, dynamics!), and is better written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathbf{F}. \tag{1}$$

To allow for relativistic mass changes, this is also often written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}, \tag{2}$$

where $\mathbf{p} \equiv m\mathbf{v}$ is the relativistic momentum; but for the most part, we'll be ignoring relativistic effects.

Newton's Second Law, together with an appropriate prescription for calculating the force $\mathbf{F}$ is often referred to as the 'equation of motion' (EOM), as it governs the time evolution of the particle's position and velocity — i.e., its movement. Solving the equation of motion in full typically requires *two* integrations, because — with the velocity itself being the time derivative of the particle's position vector r,

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \tag{3}$$

the EOM is second-order differential in time. It's not always possible to do these integrations analytically, especially if the force has some complicated dependence on space and time (as it would, for instance, if it represented the electrostatic or gravitational attraction of an ensemble of other particles). However, there do exist special circumstances where we can *always* analytically integrate the EOM at least once, thereby obtaining a closed-form expression for the velocity $\mathbf{v}$.

3.2 Conservative Forces

These special circumstances arise when the force $\mathbf{F}$ is *conservative*. In moving a particle[2] from one point $\mathbf{r}_\mathrm{a}$ to another $\mathbf{r}_\mathrm{b}$, the work done on the particle

$$W \equiv -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{4}$$

[1] From the Greek word '*dynamikos*', meaning 'powerful'

[2] In the most general sense; anything from a proton to a block of wood to a planet

by a conservative force does not depend on the route taken between the two points — only on the location of the points. An immediate corollary of this definition is that the total work done in moving the particle around a closed loop must vanish; that is,

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0. \tag{5}$$

We can use Stokes' theorem to transform the line integral in this equation into a surface integral, so that

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0, \tag{6}$$

where S is the surface bounded by the closed loop. Because this equation holdsirrespective of *which* closed loop we choose, it must be the case that

$$\nabla \times \mathbf{F} = 0 \tag{7}$$

for all conservative forces. Because the curl of a gradient is zero, this leads to result that conservative forces can always be expressed as the gradient of a scalar potential,

$$\mathbf{F} = -\nabla \phi \tag{8}$$

(the negative sign is a convention, so that the force is always directed toward lower potentials). Conversely, any force which can be expressed as the gradient of a scalar potential must be conservative.

Examples of conservative forces are the electrostatic and gravitational forces, which — for a point charge or point mass at the origin — both take the form

$$\mathbf{F}(\mathbf{r}) = \frac{A}{r^2} \mathbf{e}_r, \tag{9}$$

where A is some constant, $r \equiv |\mathbf{r}|$, and $\mathbf{e}_r$ is the unit basis vector in the radial direction at position $\mathbf{r}$. The corresponding potential is trivially found as

$$\phi(\mathbf{r}) = \frac{A}{r} + C, \tag{10}$$

where the constant of integration C is usually set to zero so that the potential goes to zero as $r \to \infty$ (this is an arbitrary but conventional choice).

An example of a *non-conservative* force is friction. When moving a particle subject to friction, $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ in eqn. (4) are antiparallel (because friction always acts oppositely to the direction of motion); hence, the net work done on the particle along *any* path is positive. In particular, the net work done around a closed loop is positive, which violates eqn. (5) — demonstrating that friction is non-conservative.

Of course, this analysis only applies at a macroscopic level. At the microscopic level, friction doesn't really exist; instead, there are just the electrostatic attractive and repulsive forces between atoms, through objects manifest the property of being solid. These forces are all conservative; but their effect is (in the case of friction) to cause small-scale, microscopic motions of the atoms (i.e., heat) rather than large-scale, macroscopic motion of the body composed by the atoms.

3.3 First Integral of the EOM

As mentioned above, an equation of motion featuring a conservative force can always be integrated at least once. To see this, we first write the acceleration in the EOM as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \sum_{i=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t}(v_i \mathbf{e}_i) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(v_i \mathbf{e}_i) \frac{\mathrm{d}x_j}{\mathrm{d}t} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_i \mathbf{e}_i}{\partial x_j} v_j \tag{11}$$

where x_i $(i = 1, 2, 3)$ are the coordinates in some arbitrary curvilinear system, and $\mathbf{e}_\mathrm{i}$ are the corresponding unit basis vectors. This can be written in the cleaner form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} (\nabla \mathbf{v}) \cdot \mathbf{v}, \tag{12}$$

where $\mathbf{x} \equiv x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ is the position vector. The term in parentheses is a rank-two tensor with components

$$(\nabla \mathbf{v})_{i,j} = \frac{\partial}{\partial x_j}(v_i \mathbf{e_i}) \tag{13}$$

when we take the dot product of this tensor with another vector, $\mathrm{d}\mathbf{x}/\mathrm{d}t$), we obtain a new vector — in eqn. (12), the acceleration vector. Although I don't want to get sidetracked into the nitty-gritty of tensor analysis, it's worth remembering the above definition of $\nabla \mathbf{v}$, as similar gradients-of-vectors will crop up later.

Substituting eqn. (12) into the equation of motion for a conservative force gives

$$m(\nabla v) \cdot \nabla v = -\nabla \phi. \tag{14}$$

This can be rewritten as

$$\frac{1}{2} m \nabla (\mathbf{v} \cdot \mathbf{v}) + \nabla \phi = 0. \tag{15}$$

This can always be integrated, to give

$$\frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) + \phi = E \tag{16}$$

where E is a constant of integration. Identifying the first term on the left-hand side as the particle kinetic energy $mv^2/2$, this equation indicates that the sum of kinetic and potential energies is a constant — that is, the total energy E is conserved.

Thus, to summarize:

- An equation of motion involving conservative forces can always be integrated at least once.
- The resulting 'first integral' is a statement of conservation of energy.

3.4 Central Forces

A *central* force is one which is always directed toward a single point in space, conveninently defined to be the origin, and moreover depends only on the distance r from this origin. Central forces are always conservative. This can be seen by expressing a generic central force as

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r, \tag{17}$$

for any function $f(r)$. This can always be derived from the potential

$$\phi(\mathbf{r}) = -\int_r^\infty f(r)\,\mathrm{d}r, \tag{18}$$

and so central forces are conservative. Note that the converse isn't necessarily true; for instance, the gravitational force from the (non-spherical) Earth is not precisely central, but it is still conservative.

In addition to satisfying conservation of energy, a system evolving under the action of a central force also conserves angular momentum.