Basic Astrophysics II

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Chapter 1

Overview

1.1 Introduction

The official course title of Astronomy 702 is 'Basic Astrophysics II' — but the real title should be 'Dynamics'. The topic of dynamics[1] concerns the time evolution of physical properties and processes. So, most of the equations we'll be encountering over the fourteen weeks of the course involve a time derivative of one form or another.

1.2 A Hierarchy of Models

Many branches of physics and astrophysics focus on phenomena occurring at a certain scale. Here, I use 'scale' in two senses — on the one hand, the physical size of the system under consideration, and on the other, the number of interacting entities (particles, planets, etc.) composing the system. Scale in the former sense will determine which forces play a dominant role, and whether a quantum or classical description is more appropriate; while scale in the latter sense dictates what sort of tools are useful for modeling the dynamics of a system. For instance, fluid (or 'continuum') models are useful in cases where there are large number of interacting particles composing a system; whereas kinetic models are the appropriate choice when only a few particles are present.

[1] From the Greek word '*dynamikos*', meaning 'powerful'

Chapter 2

Small Numbers of Particles

2.1 Individual Particles

2.1.1 The Equation of Motion

The fundamental equation governing the dynamics of all particles in the classical (non-quantum) limit is Newton's Second Law, which relates the acceleration of an individual particle to the external forces acting upon it. Although we usually learn this as the simple

$$\mathbf{F} = m\mathbf{a},$$

the equation is in fact a differential equation involving time derivatives (remember, dynamics!), and is better written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathbf{F}. \tag{2.1}$$

To allow for relativistic mass changes, this is also often written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}, \tag{2.2}$$

where $\mathbf{p} \equiv m\mathbf{v}$ is the relativistic momentum; but for the most part, we'll be ignoring relativistic effects.

Newton's Second Law, together with an appropriate prescription for calculating the force $\mathbf{F}$ is often referred to as the 'equation of motion' (EOM), as it governs the time evolution of the particle's position and velocity — i.e., its movement. Solving the equation of motion in full typically requires *two* integrations because, with the velocity itself being the time derivative of the particle's position vector r,

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \tag{2.3}$$

the EOM is second-order differential in time. It's not always possible to do these integrations analytically, especially if the force has some complicated dependence on space and time (as it would, for instance, if it represented the electrostatic or gravitational attraction of an ensemble of other particles). However, there do exist special circumstances where we can *always* analytically integrate the EOM at least once, thereby obtaining a closed-form expression for the velocity $\mathbf{v}$.

2.1.2 Conservative Forces

These special circumstances arise when the force $\mathbf{F}$ is *conservative*. In moving a particle[1] from one point $\mathbf{r}_\mathrm{a}$ to another $\mathbf{r}_\mathrm{b}$, the work done on the particle

$$W = -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{2.4}$$

[1] In the most general sense; anything from a proton to a block of wood to a planet.

by a conservative force does not depend on the route taken between the two points — only on the location of the points. An immediate corollary of this definition is that the total work done in moving the particle around a closed loop must vanish; that is,

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0. \tag{2.5}$$

We can use Stokes' theorem to transform the line integral in this equation into a surface integral, so that

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0, \tag{2.6}$$

where S is any surface bounded by the closed loop. Because this equation holds irrespective of *which* closed loop or surface we choose, it must be the case that

$$\nabla \times \mathbf{F} = 0 \tag{2.7}$$

for all conservative forces. Since the curl of a gradient is zero, this leads to result that conservative forces can always be expressed as the gradient of a scalar potential,

$$\mathbf{F} = -\nabla\phi. \tag{2.8}$$

(the negative sign is a convention, so that the force is always directed toward lower potentials). Conversely, any force which can be expressed as the gradient of a scalar must be conservative. Note that ϕ has units of energy, so although we usually refer to it as the potential, more correctly it is the potential energy.

Examples of conservative forces are the electrostatic and gravitational forces, which — for a point charge or point mass at the origin — both take the form

$$\mathbf{F}(\mathbf{r}) = \frac{A}{r^2}\mathbf{e}_r, \tag{2.9}$$

where A is some constant, $r \equiv |\mathbf{r}|$, and $\mathbf{e}_r$ is the unit basis vector in the radial direction at position $\mathbf{r}$. The corresponding potential energy is trivially found as

$$\phi(\mathbf{r}) = \frac{A}{r} + C, \tag{2.10}$$

where the constant of integration C is usually set to zero so that ϕ goes to zero as $r \to \infty$ (this is an arbitrary but conventional choice).

An example of a *non-conservative* force is friction. When moving a particle subject to friction, $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ in eqn. (2.4) are anti-parallel (because friction always acts oppositely to the direction of motion); hence, the net work done on the particle along *any* path is positive. In particular, the net work done around a closed loop is positive, which violates eqn. (2.5) — demonstrating that friction is non-conservative.

Of course, this analysis only applies at a macroscopic level. At the microscopic level, friction doesn't really exist; instead, there are just the electrostatic attractive and repulsive forces between atoms, via which objects manifest the property of being solid. These forces are all conservative; but their effect is (in the case of friction) to cause small-scale, microscopic motions of the atoms (i.e., heat) rather than large-scale, macroscopic motion.

2.1.3 First Integral of the EOM

As mentioned above, an equation of motion featuring a conservative force can always be integrated at least once. To see this, we use the chain rule to write the acceleration in the EOM as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}v_x}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}v_y}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}v_z}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}v_x}{\mathrm{d}x}v_x \\ \frac{\mathrm{d}v_y}{\mathrm{d}y}v_y \\ \frac{\mathrm{d}v_z}{\mathrm{d}z}v_z \end{pmatrix}. \tag{2.11}$$

where I've decomposed the velocity $\mathbf{v}$ into its Cartesian components v_x, v_y and v_z. The second equality can further be simplified to give

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{1}{2}\begin{pmatrix} \frac{\mathrm{d}v_x^2}{\mathrm{d}x} \\ \frac{\mathrm{d}v_y^2}{\mathrm{d}y} \\ \frac{\mathrm{d}v_z^2}{\mathrm{d}z} \end{pmatrix} = \frac{1}{2}\nabla|\mathbf{v}|^2. \tag{2.12}$$

Substituting the latter result into the EOM for a conservative force gives

$$\frac{1}{2}m\nabla|\mathbf{v}|^2 + \nabla\phi = 0. \tag{2.13}$$

This can always be integrated, to give

$$\frac{1}{2}m|\mathbf{v}|^2 + \phi = E \tag{2.14}$$

where E is a constant of integration. Identifying the first term on the left-hand side as the particle kinetic energy $mv^2/2$, this equation indicates that the sum of kinetic and potential energies is a constant — that is, the total energy E is conserved.

Thus, to summarize:

- An equation of motion involving conservative forces can always be integrated at least once.
- The resulting 'first integral' is a statement of conservation of energy.

2.1.4 Central Forces

A *central* force is one which is always directed toward a single point in space, conveniently defined to be the origin, and moreover depends only on the distance r from this origin. Central forces are always conservative. This can be seen by expressing a generic central force as

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r, \tag{2.15}$$

for any integrable function $f(r)$. This can always be derived from the potential

$$\phi(\mathbf{r}) = -\int_r^\infty f(r)\,\mathrm{d}r, \tag{2.16}$$

and so central forces are conservative. Note that the converse isn't necessarily true; for instance, the gravitational force from the (non-spherical) Earth is not precisely central, but it is still conservative.

In addition to satisfying conservation of energy, a system evolving under the action of a central force also conserves angular momentum. To demonstrate this, we first note that the initial position and velocity vectors of a particle acting under a central force serve to define a unique plane, and the force vector always lies in this plane; therefore, we need only consider the 2-dimensional motion in the plane. Using the natural choice of polar coordinates, the EOM for the particle is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta\right) - f\mathbf{e}_r = 0. \tag{2.17}$$

Here, v_r and v_θ are the velocity components in the radial and angular directions, and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the corresponding basis vectors. These basis vectors depend on the position of the particle, and therefore change with time. It can readily be demonstrated[2] that the time derivatives of the basis vectors are

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \tag{2.18}$$

and

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r, \tag{2.19}$$

[2]This phrase is often used to dodge pages and pages of tedious algebra; but in this case, it *is* pretty straightforward; just sketch how the basis vectors change over a time step δt, and take the limit $\delta t \to 0$.

where to keep things compact, I've switched to the usual dot notation to indicate derivatives with respect to time. With this switch, the velocity components become

$$v_r \equiv \dot{r} \tag{2.20}$$

and

$$v_\theta \equiv r\dot{\theta}; \tag{2.21}$$

and the EOM (2.17) becomes (after some algebra)

$$m\left[\ddot{r} - r\dot{\theta}^2 - \frac{f(r)}{m}\right]\mathbf{e}_r + m\left[r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]\mathbf{e}_\theta = 0. \tag{2.22}$$

Solutions to this equation require *both* terms in brackets to vanish. For the $\mathbf{e}_\theta$ term, we have

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0; \tag{2.23}$$

multiplying through by r, and applying a little calculus-jitsu, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0. \tag{2.24}$$

Integrating,

$$r^2\dot{\theta} = j, \tag{2.25}$$

where j is the constant of integration. This latter equation expresses conservation of angular momentum (with j the angular momentum per unit mass). It can be used to eliminate $\dot{\theta}$ from the $\mathbf{e}_r$ term in the equation of motion, to give

$$\ddot{r} - \frac{j^2}{r^3} - \frac{f(r)}{m} = 0. \tag{2.26}$$

The j-dependent term in this equation acts like an additional force, which tends to push the particle away from the origin. An analogy can be drawn here to the centrifugal force which exists in a rotating frame of reference; in both cases, the force exists to ensure that the motion of the particle in the *absence* of the central force $f(r)$ will be a straight line.

The above equation admits a first integral, and therefore conservation of energy applies even though we haven't specified the precise form of the central force; this is simply a consequence of the fact that the force is conservative. Conservation of angular momentum likewise followed from the fact that there the central force has no component in the angular direction. Depending on other properties of the force (i.e., its precise dependence on r), other conservation laws can exist.

2.2 Two Particles

2.2.1 Central Forces

Let's now move on to the case of two particles. In principle, the equation of motion for each particle will include forces due to the other particle, plus forces due to an external agent (e.g., the gravitational field of a third body). For the moment, let's ignore any external forces, and focus on the simple case of two particles interacting by central forces.

Labeling the particles using the subscripts 1 and 2, the EOMs are

$$m_1\dot{\mathbf{v}}_1 - f_{1,2}(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{2.27}$$

$$m_2\dot{\mathbf{v}}_2 + f_{2,1}(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{2.28}$$

where $f_{1,2}$ is the force exerted on particle 1 by particle 2 (positive if repulsive, negative if attractive), $f_{2,1}$ is the force exerted on particle 2 by particle 1, and $\mathbf{r}_{1,2} \equiv \mathbf{r}_1 - \mathbf{r}_2$ is the vector displacement from particle 2

to particle 1. Newton's Third Law[3] requires that $f_{2,1} = f_{1,2}$, and henceforth we shall therefore write them both simply as f. So,

$$m_1\ddot{\mathbf{r}}_1 - f(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{2.29}$$

$$m_2\ddot{\mathbf{r}}_2 + f(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0. \tag{2.30}$$

In the center-of-mass-reference frame[4], the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are anti-parallel (since the particles lie on opposite sides of the center of mass); moreover, the radial coordinates r_1 and r_2 are always related by

$$r_2 = \frac{m_1}{m_2}r_1. \tag{2.31}$$

Hence, we can represent the position vectors of *both* particles in terms of a single vector $\mathbf{r}$:

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \tag{2.32}$$

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}.$$

Substituting these expressions into eqn. (2.29) gives two identical EOMs for $\mathbf{r}$,

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\mathbf{r}} - f(r)\mathbf{e}_r = 0. \tag{2.33}$$

This describes the motion of a single particle of 'reduced mass'

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{2.34}$$

subject to the central force $f(r)$ — a problem we addressed in the preceding section.

Thus, to summarize: a system composed of two particles interacting by central forces can always be reduced to a one-body problem involving a central force.

2.3 Three Particles

2.3.1 Central Forces and the Restricted Problem

Suppose we introduce a third body into the two-body central-force problem considered above. Generally speaking, it is not possible to find a closed solution to the coupled equations of motion. However, a special case is the so-called 'restricted three-body problem', where the third particle exerts a negligible force on the first two. A good example is a test particle of vanishing mass moving in the vicinity of a binary star system.

Although the particle has no gravitational influence upon the stars (and they therefore follow standard two-body central-force dynamics), the stars *do* exert an attractive gravitational force on the particle. The equation of motion for the particle is, therefore,

$$m\ddot{\mathbf{r}} - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) - \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) = 0. \tag{2.35}$$

Here, m is the mass of the particle; M_p and M_s are the masses of the primary and secondary stars, respectively; and $\mathbf{r}_\mathrm{p}$ and $\mathbf{r}_\mathrm{s}$ are the corresponding position vectors of the stars. This equation is a little trickier to solve than it appears, because the stars are not fixed in space — rather, they orbit their common center of mass. Although we can obtain a first integral, expressing conservation of energy for the particle, it doesn't really tell us much about the particle's motion.

[3] *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*

[4] Always a good choice for multi-body problems.

However, if the stars are in circular orbits, then we can always transform to a frame of reference rotating uniformly at the orbital frequency $\Omega = 2\pi/P$. Here, P is the period of the binary system, given in terms of the orbital separation a as

$$P^2 = \frac{4\pi^2}{G(M_\mathrm{p} + M_\mathrm{s})} a^3. \tag{2.36}$$

Within this rotating frame of reference, the stars are at rest at distances

$$r_\mathrm{p} = \frac{M_\mathrm{s}}{M_\mathrm{p} + M_\mathrm{s}} a \tag{2.37}$$

and

$$r_\mathrm{s} = \frac{M_\mathrm{p}}{M_\mathrm{p} + M_\mathrm{s}} a, \tag{2.38}$$

respectively, from the center of mass (see eqn. 2.32).

But there's a wrinkle in transforming to the co-rotating frame: the acceleration must be corrected to account for the fact that this frame is not inertial. To figure out what the correction should be, note that the velocity vector in an inertial frame is related to the the corresponding velocity in the co-rotating frame by

$$\mathbf{v}_\mathrm{i} = \mathbf{v}_\mathrm{c} + \mathbf{\Omega} \times \mathbf{r}. \tag{2.39}$$

Here, $\mathbf{\Omega}$ is the angular frequency vector describing the rotation, and the 'i' and 'c' subscripts denote inertial and co-rotating, respectively. Note the absence of any subscript on the $\mathbf{r}$ term on the right-hand side. This is deliberate; although observers at rest in the inertial and co-rotating frames will disagree about the velocity vector of an object, they will both *agree* about its position vector[5]. One way of grokking this subtle point is to re-write the above equation in operator form,

$$\frac{\mathrm{d}}{\mathrm{d}t_\mathrm{i}} \mathbf{r} = \left[\frac{\mathrm{d}}{\mathrm{d}t_\mathrm{c}} + \mathbf{\Omega} \times \right] \mathbf{r} \tag{2.40}$$

The subscripts now appear on the time derivatives, to denote rates of change as measured at rest in the indicated reference frames. (As an aside: this expression applies to *any* vector, not just the position vector $\mathbf{r}$, and underscores that it is the time derivatives which change when switching frames, rather than the vector itself).

By applying the above operator *twice* to $\mathbf{r}$, we can easily obtain an expression for the relationship between the particle accelerations in the two frames:

$$\ddot{\mathbf{r}}_\mathrm{i} = \ddot{\mathbf{r}}_\mathrm{c} + 2\mathbf{\Omega} \times \dot{\mathbf{r}}_\mathrm{c} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \,. \tag{2.41}$$

Here, note the subscripts on *time derivatives* of $\mathbf{r}$, but not $\mathbf{r}$ itself! Using this expression, the equation of motion in the co-rotating frame becomes

$$m\ddot{\mathbf{r}}_\mathrm{c} + 2m\mathbf{\Omega} \times \dot{\mathbf{r}}_\mathrm{c} + m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) - \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) = 0. \tag{2.42}$$

The advantage of this equation is that, for circular orbits, the vectors $\mathbf{r}_\mathrm{p}$ and $\mathbf{r}_\mathrm{s}$ have fixed components with respect to the co-rotating frame. Comparing against the inertial equation (2.35) two new terms have appeared, corresponding to two 'fictitious' or 'inertial' forces which exist to correct for the fact that the co-rotating frame isn't inertial. The first, Coriolis term is non-zero only when the velocity $\dot{r}_\mathrm{c}$ is non-zero, whereas the second, centrifugal term is present whenever the particle is displaced from the rotation axis.

Equation (2.42) still doesn't easily yield close-form solutions, but is nevertheless quite instructive. In situations where the particle is at rest in the co-rotating frame, the net force on it is

$$\mathbf{F} = -m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) + \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) \tag{2.43}$$

[5] That said, the *components* of the position vector in each frame will differ, due to the misalignment of the coordinate axes; nevertheless, in both frames $\mathbf{r}$ points in the same direction in space, and has the same magnitude.

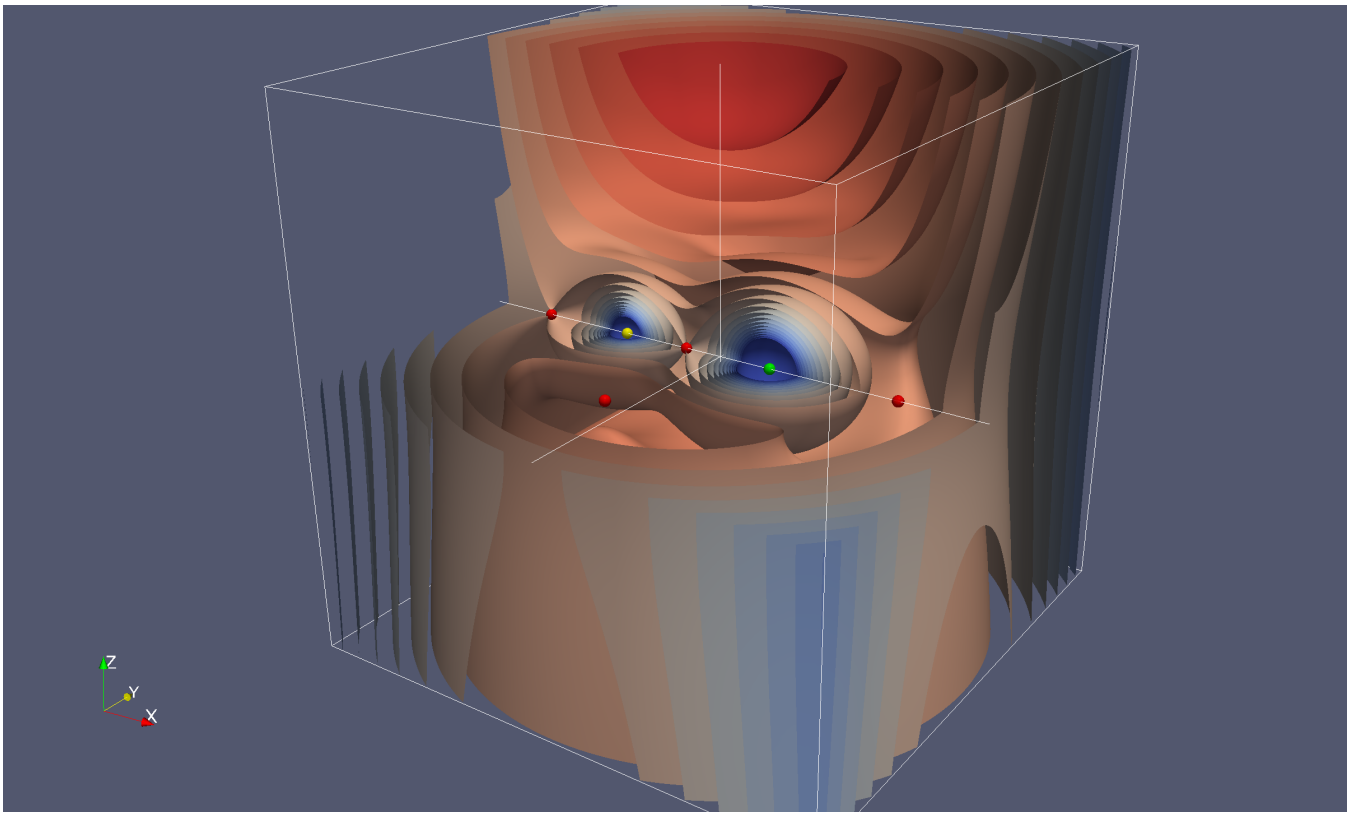

Figure 2.1: Surfaces of constant effective potential $\phi_{\rm eff}$ (cf. eqn. 2.45), for a binary system in which the primary (small green sphere, center-right) has twice the mass of the secondary (small yellow sphere, center-left). The surfaces are colored according to the effective potential value they represent (red is higher, blue is lower), and the Lagrange points are shown as red spheres.

This force can be written as the gradient of an 'effective' potential,

$$\mathbf{F} = -\nabla\phi_{\rm eff}, \tag{2.44}$$

where

$$\phi_{\rm eff} \equiv -\frac{m}{2}|\mathbf{\Omega}\times\mathbf{r}|^2 - \frac{GM_{\rm p}m}{|\mathbf{r}_{\rm p}-\mathbf{r}|} - \frac{GM_{\rm p}m}{|\mathbf{r}_{\rm p}-\mathbf{r}|} \tag{2.45}$$

combines the gravitational potential from the two stars (the second and third terms on the right-hand side) with a centrifugal potential. At points where the gradient of the effective potential vanishes, the net force on the test particle will vanish, meaning that these are points of equilibrium.

Fig. 2.1 is a 3-D rendering of the effective potential for a system in which $M_{\rm p} = 2M_{\rm s}$. Close to each of the stars, the surfaces of constant $\phi_{\rm eff}$ are nearly spherical, because one or the other of the gravitational terms in eqn. (2.45) dominates. Far from the rotation axis, the centrifugal term dominates and the equipotential surfaces are concentric cylinders. There are five points in total where $\nabla\phi_{\rm eff}$ vanishes, and these (four of which are visible in the figure) are the famous Lagrange points, traditionally labeled as $L1,\ldots,L5$. L1 through L3 lie on the line joining the two stars, with L1 between the stars, and L2 and L3 outside the stars (the convention being that L2 is closer to the smaller-mass star, and L3 to the larger-mass star). L4 and L5 lie in the orbital plane, such that they each form equilateral triangles with the two stars.

It is clear from the figure that L1, L2 and L3 are not local minima of $\phi_{\rm eff}$, but rather saddle points: the potential is *maximal* along the line joining the stars, and *minimal* along perpendicular lines. Thus, a particle at one of these points is not in stable equilibrium, and will drift away over time. How, then, do we speak of parking a satellite at one of the Lagrange points?[6] Although the points themselves are not stable,

[6]For instance, the plan is to park the *James Webb Space Telescope* at L2.

it is possible to place an object in a stable, periodic orbit near (not exactly on) of one of the points — a so-called 'halo' orbit. The orbit is maintained by the Coriolis force, which can't be modeled as an effective potential (and hence cannot be included in the definition of $\phi_{\rm eff}$).

The L4 and L5 points are local maxima of $\phi_{\rm eff}$, suggesting that they, also, are unstable. Surprisingly, however, as a particle placed at one of these points begins to drift away, the Coriolis force will send it into a stable orbit about the point, *if* the mass ratio of the system exceeds about 25 : 1. This underscores the fact that the effective potential does not tell the whole story about the motion of a test particle in the restricted three-body problem.

On a final note, as we shall see later in the course, the shapes of self-gravitating gaseous bodies — i.e., stars — are defined by equipotential surfaces (in just the same way that sea level on Earth is an equipotential surface). In binary systems, these surfaces are known as Roche lobes, and correspond exactly to the surfaces shown in Fig. 2.1. Clearly, as the radius of one star grows (e.g., due to evolutionary effects), its surface becomes progressively distorted by the other star and the centrifugal force; until eventually the surface reaches out to the L1 point, and star will begin to spill mass onto the other star. This phenomenon is 'Roche lobe overflow', and we'll be returning to it later in the course.

2.4 N Particles

2.4.1 The Virial Theorem

If dealing with three particles is tricky, what about an ensemble of N particles? Certainly, there are again no neat closed solutions; but if we are dealing with certain types of central force, we can still obtain some useful statements about the overall properties of the system once a certain type of equilibrium has been reached.

To see this, consider the scalar moment of inertia of the ensemble,

$$I = \sum_i m_i |\mathbf{r}_i|^2, \tag{2.46}$$

where i indexes the particle under consideration, and the sum is taken over all particles, $i = 1, \ldots, N$. Taking the time derivative,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\sum_i \mathbf{p}_i \cdot \mathbf{r}_i \equiv 2G \tag{2.47}$$

where $\mathbf{p}_i \equiv m_i \mathbf{v}_i$ is the momentum of particle i, and the second equality defines the so-called 'virial[7] of Clausius G.

Things get interesting if we take the time derivative one more time, to find

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \left[m_i \ddot{\mathbf{r}} \cdot \mathbf{r} + m_i \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \right]. \tag{2.48}$$

The second term in square brackets can be recognized as twice the total kinetic energy

$$K \equiv \frac{1}{2} \sum_i m_i |\dot{\mathbf{r}}|^2 \tag{2.49}$$

of the ensemble, so

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i m_i \ddot{\mathbf{r}} \cdot \mathbf{r} + 2K. \tag{2.50}$$

To handle the remaining summation, we make use of Newton's second law to write

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \tag{2.51}$$

where $\mathbf{F}_i$ is the net force on the i'th particle. Hence, we have

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \mathbf{F}_i \cdot \mathbf{r} + 2K, \tag{2.52}$$

[7] The word *virial* comes from the Latin *vis*, meaning force or energy

which is an exact (although still rather un-illuminating) result.

What happens when we average the time derivative of the virial over some time interval τ? Clearly,

$$\left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle \equiv \frac{1}{\tau} \int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\, \mathrm{d}t = \frac{G(\tau) - G(0)}{\tau}. \tag{2.53}$$

There are two cases where this time average might evaluate to zero. First, if the system is strictly periodic, such that it returns to its initial state after time τ, then the numerator on the right-hand side will vanish. Second, if we are dealing with a stably bound system, then we can expect the numerator to remain finite; then, over a suitably long time (i.e., $\tau \to \infty$), it should be the case that $\langle \mathrm{d}G/\mathrm{d}t \rangle \to 0$. If either of these cases hold, then eqn. (2.52) becomes

$$\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle + 2\langle K \rangle = 0, \tag{2.54}$$

which is the renowned *Virial Theorem*.

Suppose the force $\mathbf{F}_i$ can be decomposed into the sum of individual forces arising from pair-wise interactions with other particles; that is,

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij}, \tag{2.55}$$

where $\mathbf{F}_{ij}$ denotes the force on the i'th particle due to the j'th particle. Then, the virial theorem becomes

$$\left\langle \sum_i \sum_{j \neq i} \mathbf{F}_{ij} \cdot \mathbf{r}_i \right\rangle + 2\langle K \rangle = 0. \tag{2.56}$$

If we write out this double summation in full, we can always pair up terms involving $\mathbf{F}_{i,j}$ and $\mathbf{F}_{i,j}$, for any choice of i and j (so long as $i \neq j$). By Newton's third law, $\mathbf{F}_{i,j} = -\mathbf{F}_{j,i}$, and so the virial theorem becomes

$$\frac{1}{2} \left\langle \sum_i \sum_{j \neq i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j) \right\rangle + 2\langle K \rangle = 0, \tag{2.57}$$

where the factor of $1/2$ corrects for the fact that a summation over *pairs* will end up double counting.

Let's now focus on a case of particular astrophysical interest: forces arising due to mutual gravitational interactions,

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} (\mathbf{r}_i - \mathbf{r}_j). \tag{2.58}$$

Substituting this into eqn. (2.57), we find after a little algebra that

$$-\frac{1}{2} \left\langle \sum_i \sum_{j \neq i} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \right\rangle + 2\langle K \rangle = 0. \tag{2.59}$$

The summation term can be recognized immediately as the gravitational potential energy of the system,

$$U = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{2.60}$$

where, again, the factor $1/2$ appears to correct for double counting. So, the final result is

$$\langle U \rangle + 2\langle K \rangle = 0, \tag{2.61}$$

which is the virial theorem for a self-gravitating system.

This result looks rather trivial, but in fact is quite profound. If we consider the *total* energy for the system,

$$E = U + V \tag{2.62}$$

, then it must be the case that

$$E = -\langle K \rangle = \frac{1}{2} \langle U \rangle. \tag{2.63}$$

A few points to note here:

- The total energy is negative; this is consistent with our requirement that the system is bound.
- As the system loses energy (i.e., E becomes more negative), $\langle U \rangle$ must become more negative also — i.e., the system contracts.
- Likewise, as the system loses energy, $\langle K \rangle$ must become more positive — i.e., the system becomes 'faster' or 'hotter'.

Chapter 3

Large Numbers of Particles

3.1 The Fluid Picture

At the most basic level, a fluid[1] is any substance which can continuously deform (i.e., flow) under an applied shear stress. Liquids are the most obvious examples of fluids, but gases also fall into this category, as do certain esoteric substances such as silly putty (classified as a viscoelastic fluid). Indeed, the distinction between a fluid and a solid is not always clear cut, and finding a rigorous definition can require quite a bit of effort.

With this in mind, at this stage I'm not going to formally define what a fluid is — instead, I'm going to invoke our experiences and intuition concerning everyday fluids such as water, oil and air. When describing these substances, we don't think in terms of the positions and velocities of individual atoms and molecules making up the fluid (i.e., microscopic properties). Rather, we view the fluid as a *continuum* characterized by macroscopic properties such as density $\rho(\mathbf{r}, t)$, velocity $\mathbf{v}(\mathbf{r}, t)$ and pressure $p(\mathbf{r}, t)$, all of which are functions of location in the fluid $\mathbf{r}$ and time t; that is, the fluid is described by a set of scalar and vector fields[2]. The time evolution of these fields is governed by a set of partial differential equations known variously as the fluid equations, the hydrodynamic equations (or 'hydro equations' for short), the Navier-Stokes equations and/or the Euler equations. We'll derive these equations in the following sections.

3.2 Fluid Mass Conservation

3.2.1 Integral Form

Consider a finite volume of space $\mathcal{V}$, which can in principle have any shape or size. Suppose this 'control volume' is fixed in space; then, as fluid flows through the walls of the volume, the total mass contained,

$$\mathcal{M} = \int_{\mathcal{V}} \rho \mathrm{d}\tau, \tag{3.1}$$

(where $\mathrm{d}\tau$ is the differential volume element) will evolve over time. More specifically, because mass must be conserved, the rate of change of $\mathcal{M}$ must balance the rate at which mass is entering or leaving the control volume through its boundaries. For a boundary element $\mathrm{d}\mathbf{S}$[3] of $\mathcal{V}$, the amount of mass passing *out of* $\mathcal{V}$ per second is

$$\mathrm{d}\mathcal{M} = \rho \mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{3.2}$$

To understand this expression, note that the *volume* of material passing through $\mathrm{d}\mathbf{S}$ per second is $\mathbf{v} \cdot \mathrm{d}\mathbf{S}$; multiplying by the density of this material then gives the mass flowing through per second.

[1] From the Latin *fluidus*, that which flows.

[2] 'Field' in the most general sense — a physical quantity associated with each point of spacetime.

[3] Defined in the usual manner; the magnitude of $\mathrm{d}\mathbf{S}$ is the area of the element, and its direction is normal to the boundary and pointing outward.

If we integrate over all elements composing the boundary surface $\mathcal{S}$ of the control volume, then we end up with an expression for the change in the mass contained in $\mathcal{V}$ per second,

$$\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = -\int_{\mathcal{S}} \rho \mathbf{v} \cdot \mathrm{d}\mathbf{S} \tag{3.3}$$

(here, the minus sign appears because the expression derived above was for the mass lost from the volume). Replacing $\mathcal{M}$ by its definition (3.1, and bringing the right-hand side over to the left, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}} \rho \, \mathrm{d}\tau\right] + \int_{\mathcal{S}} \rho \mathbf{v} \cdot \mathrm{d}\mathbf{S} = 0. \tag{3.4}$$

Since the boundaries of $\mathcal{V}$ are fixed (i.e., they don't change with time), we can bring the time derivative inside the volume integral, to find

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\tau + \int_{\mathcal{S}} \rho \mathbf{v} \cdot \mathrm{d}\mathbf{S} = 0 \tag{3.5}$$

(note that the time derivative now becomes a partial derivative, because ρ is a function of time *and* space). This is one of the hydro equations — the *mass conservation equation* — expressed in integral form. A variant on this equation, which is conceptually useful, is

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\tau + \int_{\mathcal{S}} \mathbf{F}_\rho \cdot \mathrm{d}\mathbf{S} = 0; \tag{3.6}$$

here,

$$\mathbf{F}_\rho \equiv \rho \mathbf{v} \tag{3.7}$$

is the mass flux vector, which specifies the mass flow per unit area (perpendicular to the vector) per unit time. When the mass conservation equation is written in terms of $\mathbf{F}_\rho$, it becomes clear how the changes in the mass within $\mathcal{V}$ are due to the flow of mass through $\mathcal{S}$.

3.2.2 Differential Form

Equations (3.5 and 3.6) are the most generally valid form for the mass conservation equation (also known as the continuity equation). A somewhat more restrictive form, but also one that can be more practically useful (since it doesn't require us to consider a finite control volume), can be derived if we assume that the fluid has no discontinuities in the density or the velocity. Then, the spatial derivatives of these latter quantities are well defined, and we can make use of the divergence theorem to rewrite the surface integral in eqn. (3.6) as a volume integral,

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\tau + \int_{\mathcal{V}} \nabla \cdot \mathbf{F}_\rho \, \mathrm{d}\tau = 0. \tag{3.8}$$

Over a sufficiently small volume, the integrands in this equation can be assumed constant, and therefore it must be the case that at each point in space,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F}_\rho = 0. \tag{3.9}$$

This is again the continuity equation, but now expressed in differential form. Using the definition (3.7) of the mass flux, we can also write

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.10}$$

3.2.3 Alternative Lagrangian Derivation

The derivations presented above are from the standpoint of a fixed control volume, through which the fluid flows. This is known as an 'Eulerian' framework. An alternative way of arriving at the fluid equations is a 'Lagrangian' framework, in which the control volume $\mathcal{V}$ is assumed to move with the fluid. The mass contained within $\mathcal{V}$ is

$$\mathcal{M} = \int_{\mathcal{V}(t)} \rho \mathrm{d}\tau; \tag{3.11}$$

this almost identical to eqn. (3.11), but the volume now has an explicit time dependence, indicated as $\mathcal{V}(t)$.

Because the control volume moves with the fluid, no material crosses its boundaries[4], and therefore $\mathcal{M}$ must remain constant:

$$\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{V}(t)} \rho \mathrm{d}\tau\right] = 0. \tag{3.12}$$

Can we simplify the above expression by taking the time derivative under the integral sign, as we did before? The answer is generally no, because $\mathcal{V}(t)$ now depends on the time. However, we *can* make use of a mathematical result, the Reynolds Transport Theorem, which — for any quantity f — expands out the time derivative of a volume integral as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} f \,\mathrm{d}\tau = \int_{\mathcal{V}(t)} \frac{\partial f}{\partial t}\,\mathrm{d}\tau + \int_{\mathcal{S}(t)} f\mathbf{v}\cdot\mathrm{d}\mathbf{S}. \tag{3.13}$$

Setting $f = \rho$, and combining this with eqn. (3.12), we obtain

$$\int_{\mathcal{V}(t)} \frac{\partial \rho}{\partial t}\,\mathrm{d}\tau + \int_{\mathcal{S}(t)} \rho\mathbf{v}\cdot\mathrm{d}\mathbf{S} = 0. \tag{3.14}$$

This is essentially identical to eqn. (3.5), apart from the appearance of $\mathcal{V}(t)$ instead of $\mathcal{V}$, and likewise for $\mathcal{S}(t)$. If the density and velocity are continuous, then the divergence theorem can be used as before to convert this into differential equation identical to eqn. (3.10).

3.2.4 The Advective Derivative

The Reynolds Transport Theorem can be used to derive another useful result. Setting $f = 1$ in eqn. (3.12), we obtain an expression for the volume rate of change of the moving control volume,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \mathrm{d}\tau \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \int_{\mathcal{S}(t)} \mathbf{v}\cdot\mathrm{d}\mathbf{S}. \tag{3.15}$$

If we assume as before that the fluid has no discontinuities, then the divergence theorem gives

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \int_{\mathcal{V}} \nabla\cdot\mathbf{v}\,\mathrm{d}\tau. \tag{3.16}$$

In an *incompressible* fluid, the left-hand side of this expression vanishes — although any arbitrary control volume may become distorted as it is carried along with the incompressible fluid, its overall volume remains constant. This means that an incompressible fluid must satisfy

$$\nabla\cdot\mathbf{v} = 0 \tag{3.17}$$

everywhere. Combining this restriction with eqn. (3.10), the continuity equation for an incompressible fluid becomes

$$\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = 0. \tag{3.18}$$

This can also be written

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{3.19}$$

where we have introduced the differential operator

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla. \tag{3.20}$$

This time-derivative operator is known by many names: the advective derivative, the material derivative, the Lagrangian derivative, the Stokes derivative. When applied to any variable, it gives the temporal rate of change of that variable as experienced by a test particle (or small control volume) which moves with the flow. So, eqn. (3.19) tells us that the density experienced by a test particle moving with the flow must be constant, if the fluid is incompressible.

[4] At least, at the macroscopic level; there are still microscopic motions across the boundary, but with no net effect on the fluid properties.

3.3 Fluid Momentum Conservation

3.3.1 Integral Form

The next hydro equation we're going to derive describes the conservation of momentum; so, this will be the fluid version of Newton's second law, a.k.a. the equation of motion. In an Eulerian framework, the total momentum of a fixed control volume is

$$\mathcal{P} = \int_{\mathcal{V}} \rho \mathbf{v} \, \mathrm{d}\tau; \tag{3.21}$$

that is, the volume integral of the momentum per unit volume $\rho\mathbf{v}$. As in the preceding section, the rate of change of $\mathcal{P}$ must balance the rate at which momentum is entering or leaving the control volume through its boundaries. The transfer of momentum into or out of the volume happens at two levels — macroscopically due to the actual fluid flow, but also microscopically due to pressure forces. Thus, we write

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho\mathbf{v}) \, \mathrm{d}\tau = \left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{macro}} + \left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{micro}}. \tag{3.22}$$

The macroscopic transfer rate is given, analogously to eqn. (3.3), by

$$\left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{macro}} = -\int_{\mathcal{S}} \rho\mathbf{v}\mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{3.23}$$

In much the same way we defined the mass flux vector $\mathbf{F}_\rho$ (cf. eqn. 3.7), we can define the momentum flux *dyad* as

$$\mathsf{F}_{\rho\mathbf{v}} \equiv \rho\mathbf{v}\mathbf{v}. \tag{3.24}$$

A dyad is formed via a special kind of product — the dyadic product[5] — between a pair of vectors. Dyadic notation is a simplified approach to dealing with rank-2 tensors; it is discussed more fully in the supplementary notes (*Vector and Dyadic Analysis*) I've provided online. Some highlights:

- $\mathbf{ab} \neq \mathbf{ba}$ — the dyadic product is not commutative.
- $(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ — the dyadic and scalar product are associative.
- The scalar product between a dyad and a vector is another vector; this follows directly from the associative property above.
- The scalar product between two dyads is another dyad; this again follows from the associative property.
- There exists an identity dyad, I, for which $\mathsf{I} \cdot \mathbf{a} = \mathbf{a}$ for any vector $\mathbf{a}$, and $\mathsf{I} \cdot \mathsf{A} = \mathsf{A}$ for any dyad A.

With the definition above of $\mathsf{F}_{\rho\mathbf{v}}$, the macroscopic momentum transfer rate in/out of the control volume becomes

$$\left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{macro}} = -\int_{\mathcal{S}} \mathsf{F}_{\rho\mathbf{v}} \cdot \mathrm{d}\mathbf{S}. \tag{3.25}$$

In the absence of viscosity (which we're neglecting for the moment), the corresponding microscopic transfer rate is given by the net effect of pressure forces acting on the boundary $\mathcal{S}$ of the control volume (recall that a force is entirely equivalent to the rate of change of volume). The force exerted on a surface element $\mathrm{d}\mathbf{S}$ by a pressure p is $-p\,\mathrm{d}\mathbf{S}$[6]. Integrating over the whole of $\mathcal{S}$, we obtain the microscopic momentum transfer rate

$$\left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{micro}} = -\int_{\mathcal{S}} p \, \mathrm{d}\mathbf{S}. \tag{3.26}$$

Defining the dyadic pressure as

$$\mathsf{P} = p\,\mathsf{I}, \tag{3.27}$$

[5] Known in tensor algebra as the 'outer product'.

[6] That is, the force is directed normal to the surface and *inward*, and has magnitude $p\,\mathrm{d}S$.

where $\mathbf{I}$ is the identity dyad introduced previously, we can alternatively write

$$\left.\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}\right|_{\text{micro}} = -\int_{\mathcal{S}} \mathbf{P}\cdot\mathrm{d}\mathbf{S}. \tag{3.28}$$

Combining this expression with eqns. (3.22) and (3.25), we arrive at the result

$$\int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho\mathbf{v})\,\mathrm{d}\tau + \int_{\mathcal{S}} (\mathbf{F}_{\rho\mathbf{v}} + \mathbf{P})\cdot\mathrm{d}\mathbf{S} = 0 \tag{3.29}$$

This is the fluid momentum conservation equation, written in integral form.

3.3.2 Differential Form

As before, if there are no discontinuities in the fluid, then we can also derive a differential version of the equation. Here, we make use of the divergence theorem for any dyadic quantity $\mathbf{A}$,

$$\int_{\mathcal{S}} \mathbf{A}\cdot\mathrm{d}\mathbf{S} = \int_{\mathcal{V}} \nabla\cdot\mathbf{A}\,\mathrm{d}\tau. \tag{3.30}$$

(here, note that the divergence of $\mathbf{A}$ is a vector). Applying this to the conservation equation, it follows that

$$\int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho\mathbf{v})\,\mathrm{d}\tau + \int_{\mathcal{V}} \nabla\cdot(\mathbf{F}_{\rho\mathbf{v}} + \mathbf{P})\,\mathrm{d}\tau = 0. \tag{3.31}$$

Over a sufficiently small volume, the integrands in this equation can be assumed constant, and therefore it must be the case that at each point in space,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\mathbf{F}_{\rho\mathbf{v}} + \mathbf{P}) = 0. \tag{3.32}$$

Expanding out the momentum flux dyad,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + p\mathbf{I}) = 0. \tag{3.33}$$

This is the fluid momentum conservation equation, written in differential form.

Other forms of this equation can be derived by expanding out the divergence term. From eqn. (A.142) of *Vector and Dyadic Analysis*,

$$\nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \nabla\cdot(\rho\mathbf{v})\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} \tag{3.34}$$

(this is just the chain rule), and likewise from eqn. (A.156),

$$\nabla\cdot(p\,\mathbf{I}) = \nabla p. \tag{3.35}$$

Therefore, the momentum conservation equation becomes

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v})\mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0. \tag{3.36}$$

Combining this with the continuity equation (3.10), we obtain

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0; \tag{3.37}$$

or, using the material time derivative introduced above in (3.20),

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \nabla p = 0, \tag{3.38}$$

which is basically Newton's second law for a control volume being acted upon by pressure forces.