Basic Astrophysics II

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1 Introduction

The official course title of Astronomy 702 is 'Basic Astrophysics II' — but the real title should be 'Dynamics'. The topic of dynamics[1] concerns the time evolution of physical properties and processes. So, most of the equations we'll be encountering over the 14 weeks of the course involve a time derivative of one form or another.

2 A Hierarchy of Models

Many branches of physics and astrophysics focus on phenomena occurring at a certain scale. Here, I use 'scale' in two senses — on the one hand, the physical size of the system under consideration, and on the other, the number of interacting entities (particles, planets, etc.) composing the system. Scale in the former sense will determine which forces play a dominant role, and whether a quantum or classical description is more appropriate; while scale in the latter sense dictates what sort of tools are useful for modeling the dynamics of a system. For instance, fluid (or 'continuum') models are useful in cases where there are large number of interacting particles composing a system; whereas kinetic models are the appropriate choice when only a few particles are present. The latter are generally

3 Individual Particles

3.1 The Equation of Motion

The fundamental equation governing the dynamics of all particles in the classical (non-quantum) limit is Newton's Second Law, which relates the acceleration of an individual particle to the external forces acting upon it. Although we usually learn this as the simple

$$\mathbf{F} = m\mathbf{a},$$

the equation is in fact a differential equation involving time derivatives (remember, dynamics!), and is better written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathbf{F}. \tag{1}$$

[1] From the Greek word '*dynamikos*', meaning 'powerful'

To allow for relativistic mass changes, this is also often written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}, \tag{2}$$

where $\mathbf{p} \equiv m\mathbf{v}$ is the relativistic momentum; but for the most part, we'll be ignoring relativistic effects.

Newton's Second Law, together with an appropriate prescription for calculating the force $\mathbf{F}$ is often referred to as the 'equation of motion' (EOM), as it governs the time evolution of the particle's position and velocity — i.e., its movement. Solving the equation of motion in full typically requires *two* integrations, because — with the velocity itself being the time derivative of the particle's position vector r,

$$\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \tag{3}$$

the EOM is second-order differential in time. It's not always possible to do these integrations analytically, especially if the force has some complicated dependence on space and time (as it would, for instance, if it represented the electrostatic or gravitational attraction of an ensemble of other particles). However, there do exist special circumstances where we can *always* analytically integrate the EOM at least once, thereby obtaining a closed-form expression for the velocity $\mathbf{v}$.

3.2 Conservative Forces

These special circumstances arise when the force $\mathbf{F}$ is *conservative*. In moving a particle[2] from one point $\mathbf{r}_\mathrm{a}$ to another $\mathbf{r}_\mathrm{b}$, the work done on the particle

$$W \equiv -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{4}$$

by a conservative force does not depend on the route taken between the two points — only on the location of the points. An immediate corollary of this definition is that the total work done in moving the particle around a closed loop must vanish; that is,

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0. \tag{5}$$

We can use Stokes' theorem to transform the line integral in this equation into a surface integral, so that

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0, \tag{6}$$

where S is the surface bounded by the closed loop. Because this equation holds irrespective of *which* closed loop we choose, it must be the case that

$$\nabla \times \mathbf{F} = 0 \tag{7}$$

for all conservative forces. Because the curl of a gradient is zero, this leads to result that conservative forces can always be expressed as the gradient of a scalar potential,

$$\mathbf{F} = -\nabla\phi \tag{8}$$

(the negative sign is a convention, so that the force is always directed toward lower potentials). Conversely, any force which can be expressed as the gradient of a scalar potential must be conservative.

Examples of conservative forces are the electrostatic and gravitational forces, which — for a point charge or point mass at the origin — both take the form

$$\mathbf{F}(\mathbf{r}) = \frac{A}{r^2}\mathbf{e}_r, \tag{9}$$

where A is some constant, $r \equiv |\mathbf{r}|$, and $\mathbf{e}_r$ is the unit basis vector in the radial direction at position $\mathbf{r}$. The corresponding potential is trivially found as

$$\phi(\mathbf{r}) = \frac{A}{r} + C, \tag{10}$$

[2] In the most general sense; anything from a proton to a block of wood to a planet

where the constant of integration C is usually set to zero so that the potential goes to zero as $r \to \infty$ (this is an arbitrary but conventional choice).

An example of a *non-conservative* force is friction. When moving a particle subject to friction, $\mathbf{F}$ and $\mathrm{d}\mathbf{r}$ in eqn. (4) are anti-parallel (because friction always acts oppositely to the direction of motion); hence, the net work done on the particle along *any* path is positive. In particular, the net work done around a closed loop is positive, which violates eqn. (5) — demonstrating that friction is non-conservative.

Of course, this analysis only applies at a macroscopic level. At the microscopic level, friction doesn't really exist; instead, there are just the electrostatic attractive and repulsive forces between atoms, through objects manifest the property of being solid. These forces are all conservative; but their effect is (in the case of friction) to cause small-scale, microscopic motions of the atoms (i.e., heat) rather than large-scale, macroscopic motion of the body composed by the atoms.

3.3 First Integral of the EOM

As mentioned above, an equation of motion featuring a conservative force can always be integrated at least once. To see this, we use the chain rule to write the acceleration in the EOM as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}v_x}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}v_y}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}v_z}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}v_x}{\mathrm{d}x}v_x \\ \frac{\mathrm{d}v_y}{\mathrm{d}y}v_y \\ \frac{\mathrm{d}v_z}{\mathrm{d}z}v_z \end{pmatrix}. \tag{11}$$

where I've decomposed the velocity $\mathbf{v}$ into its Cartesian components v_x, v_y and v_z. The second equality can further be simplified to give

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{1}{2}\begin{pmatrix} \frac{\mathrm{d}v_x^2}{\mathrm{d}x} \\ \frac{\mathrm{d}v_y^2}{\mathrm{d}y} \\ \frac{\mathrm{d}v_z^2}{\mathrm{d}z} \end{pmatrix} = \frac{1}{2}\nabla|\mathbf{v}|^2. \tag{12}$$

Substituting the latter result into the equation of motion for a conservative force gives

$$\frac{1}{2}m\nabla|\mathbf{v}|^2 + \nabla\phi = 0. \tag{13}$$

This can always be integrated, to give

$$\frac{1}{2}m|\mathbf{v}|^2 + \phi = E \tag{14}$$

where E is a constant of integration. Identifying the first term on the left-hand side as the particle kinetic energy $mv^2/2$, this equation indicates that the sum of kinetic and potential energies is a constant — that is, the total energy E is conserved.

Thus, to summarize:

- An equation of motion involving conservative forces can always be integrated at least once.
- The resulting 'first integral' is a statement of conservation of energy.

3.4 Central Forces

A *central* force is one which is always directed toward a single point in space, conveniently defined to be the origin, and moreover depends only on the distance r from this origin. Central forces are always conservative. This can be seen by expressing a generic central force as

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r, \tag{15}$$

for any function $f(r)$. This can always be derived from the potential

$$\phi(\mathbf{r}) = -\int_r^\infty f(r)\,\mathrm{d}r, \tag{16}$$

and so central forces are conservative. Note that the converse isn't necessarily true; for instance, the gravitational force from the (non-spherical) Earth is not precisely central, but it is still conservative.

In addition to satisfying conservation of energy, a system evolving under the action of a central force also conserves angular momentum. To demonstrate this, we first note that the initial position and velocity vectors of a particle acting under a central force serve to define a unique plane, and the force vector always lies in this plane; therefore, we need only consider the 2-dimensional motion in the plane. Using the natural choice of polar coordinates, the equation of motion for the particle is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta\right) - f\mathbf{e}_r = 0. \tag{17}$$

Here, v_r and v_θ are the velocity components in the radial and angular directions, and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the corresponding basis vectors. These basis vectors depend on the position of the particle, and therefore change with time. It can readily be demonstrated[3] that the time derivatives of the basis vectors are

$$\dot{\mathbf{e}_r} = \dot\theta \mathbf{e}_\theta \tag{18}$$

and

$$\dot{\mathbf{e}_\theta} = -\dot\theta \mathbf{e}_r, \tag{19}$$

where to keep things compact, I've switched to the usual dot notation to indicate derivatives with respect to time. With this switch, the velocity components become

$$v_r \equiv \dot r \tag{20}$$

and

$$v_\theta \equiv r\dot\theta; \tag{21}$$

and the equation of motion (17) becomes (after some algebra)

$$m\left[\ddot r - r\dot\theta - f(r)\right]\mathbf{e}_r + \left[r\ddot{theta} + \dot r\dot\theta + r\dot\theta^2\right]\mathbf{e}_\theta = 0. \tag{22}$$

Solutions to this equation require *both* terms in brackets to vanish. For the $\mathbf{e}_\theta$ term, we have

$$r\ddot{theta} + \dot r\dot\theta + \dot r\dot\theta = 0; \tag{23}$$

multiplying through by r, and applying a little calculus-jitsu, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\theta) = 0. \tag{24}$$

Integrating,

$$r^2\dot\theta = j, \tag{25}$$

where j is the constant of integration. This latter equation expresses conservation of angular momentum (with j the angular momentum per unit mass). It can be used to eliminate $\dot\theta$ from the $\mathbf{e}_r$ term in the equation of motion, to give

$$\ddot r - \frac{j}{r} - f(r) = 0. \tag{26}$$

The j-dependent term in this equation acts like an additional force, which tends to push the particle away from the origin. An analogy can be drawn here to the centrifugal force which exists in a rotating frame of reference; in both cases, the force exists to ensure that the motion of the particle in the *absence* of the central force $f(r)$ will be a straight line.

The above equation admits a first integral, and therefore conservation of energy applies even though we haven't specified the precise form of the central force; this is simply a consequence of the fact that the force is conservative. Conservation of angular momentum likewise followed from the fact that there the central force has no component in the angular direction. Depending on other properties of the force (i.e., its precise dependence on r), other conservation laws can exist.

[3] This phrase is often used to dodge pages and pages of tedious algebra; but in this case, it *is* pretty straightforward; just sketch how the basis vectors change over a time step δt, and take the limit $\delta t \to 0$.

4 Two Particles

4.1 Central Forces

Let's now move on to the case of two particles. In principle, the equation of motion for each particle will include forces due to the other particle, plus forces due to an external agent (e.g., the gravitational field of a third body). For the moment, let's ignore any external forces, and focus on the simple case of two particles interacting by central forces.

Labeling the particles using the subscripts 1 and 2, the equations of motion are

$$\dot{\mathbf{v}}_1 - f_{1,2}(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{27}$$

$$\dot{\mathbf{v}}_2 + f_{2,1}(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{28}$$

where $f_{1,2}$ is the force exerted on particle 1 by particle 2 (positive if repulsive, negative if attractive), $f_{2,1}$ is the force exerted on particle 2 by particle 1, and $\mathbf{r}_{1,2} \equiv \mathbf{r}_1 - \mathbf{r}_2$ is the vector displacement from particle 2 to particle 1. Newton's Third Law[4] requires that $f_{2,1} = f_{1,2}$, and henceforth we shall therefore write them both simply as f. So,

$$m_1\ddot{r}_1 - f(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0, \tag{29}$$

$$m_2\ddot{r}_2 + f(r_{1,2})\frac{\mathbf{r}_{1,2}}{r_{1,2}} = 0. \tag{30}$$

In the center-of-mass-reference frame[5], the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are anti-parallel (since the particles lie on opposite sides of the center of mass); moreover, the radial coordinates r_1 and r_2 are always related by

$$r_2 = \frac{m_1}{m_2}r_1. \tag{31}$$

Hence, we can represent the position vectors of *both* particles in terms of a single vector $\mathbf{r}$:

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \tag{32}$$

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{33}$$

Substituting these expressions into eqn. (29) gives two identical equations of motion for $\mathbf{r}$,

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{r} - f(r)\frac{\mathbf{r}}{r} = 0. \tag{34}$$

This describes the motion of a single particle of 'reduced mass'

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{35}$$

subject to the central force $f(r)$ — a problem we addressed in the preceding section.

Thus, to summarize: a system composed of two particles interacting by central forces can always be reduced to a one-body problem involving a central force.

5 Three Particles

5.1 Central Forces and the Restricted Problem

Suppose we introduce a third body into the two-body central-force problem considered above. Generally speaking, it is not possible to find a closed solution to the coupled equations of motion. However, a special

[4] Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.

[5] Always a good choice for multi-body problems.

case is the so-called 'restricted three-body problem', where the third particle exerts a negligible force on the first two. A good example is a test particle of vanishing mass moving in the vicinity of a binary star system.

Although the particle has no gravitational influence upon the stars (and they therefore follow standard two-body central-force dynamics), the stars *do* exert an attractive gravitational force on the particle. The equation of motion for the particle is, therefore,

$$m\ddot{\mathbf{r}} - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) - \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) = 0. \tag{36}$$

Here, m is the mass of the particle; M_p and M_s are the masses of the primary and secondary stars, respectively; and $\mathbf{r}_\mathrm{p}$ and $\mathbf{r}_\mathrm{s}$ are the corresponding position vectors of the stars. This equation is a little trickier to solve than it appears, because the stars are not fixed in space — rather, they orbit their common center of mass. Although we can obtain a first integral, expressing conservation of energy for the particle, it doesn't really tell us much about the particle's motion.

However, if the stars are in circular orbits, then we can always transform to a frame of reference rotating uniformly at the orbital frequency $\Omega = 2\pi/P$. Here, P is the period of the binary system, given in terms of the orbital separation a as

$$P^2 = \frac{4\pi^2}{G(M_\mathrm{p} + M_\mathrm{s})}a^3. \tag{37}$$

Within this rotating frame of reference, the stars are at rest at distances

$$r_\mathrm{p} = \frac{M_\mathrm{s}}{M_\mathrm{p} + M_\mathrm{s}}a \tag{38}$$

and

$$r_\mathrm{s} = \frac{M_\mathrm{p}}{M_\mathrm{p} + M_\mathrm{s}}a, \tag{39}$$

respectively, from the center of mass (see eqn. 32).

But there's a wrinkle in transforming to the co-rotating frame: the acceleration must be corrected to account for the fact that this frame is not inertial. To figure out what the correction should be, note that the velocity vector in an inertial frame is related to the the corresponding velocity in the co-rotating frame by

$$\mathbf{v}_\mathrm{i} = \mathbf{v}_\mathrm{c} + \boldsymbol{\Omega} \times \mathbf{r}. \tag{40}$$

Here, $\boldsymbol{\Omega}$ is the angular frequency vector describing the rotation, and the i and c subscripts denote inertial and co-rotating, respectively. Note the absence of any subscript on the $\mathbf{r}$ term on the right-hand side. This is deliberate; although observers at rest in the inertial and co-rotating frames will disagree about the velocity vector of an object, they will both *agree* about its position vector[6]. One way of grokking this subtle point is to re-write the above equation in operator form,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_\mathrm{i}\mathbf{r} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_\mathrm{c} + \boldsymbol{\Omega}\times\right]\mathbf{r} \tag{41}$$

The subscripts now appear on the time derivatives, to denote rates of change as measured at rest in the indicated reference frames. (As an aside: this expression applies to *any* vector, not just the position vector $\mathbf{r}$, and underscores that it is the time derivatives which change when switching frames, rather than the vector itself).

By applying the above operator *twice* to $\mathbf{r}$, we can easily obtain an expression for the relationship between the particle accelerations in the two frames:

$$\ddot{\mathbf{r}}_\mathrm{i} = \ddot{\mathbf{r}}_\mathrm{c} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}}_\mathrm{c} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\,. \tag{42}$$

Here, note the subscripts on *time derivatives* of $\mathbf{r}$, but not $\mathbf{r}$ itself! Using this expression, the equation of motion in the co-rotating frame becomes

$$m\ddot{\mathbf{r}}_\mathrm{c} + 2m\boldsymbol{\Omega} \times \dot{\mathbf{r}}_\mathrm{c} + m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) - \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) = 0. \tag{43}$$

[6]That said, the *components* of the position vector in each frame will differ, due to the misalignment of the coordinate axes; nevertheless, in both frames $\mathbf{r}$ points in the same direction in space, and has the same magnitude.

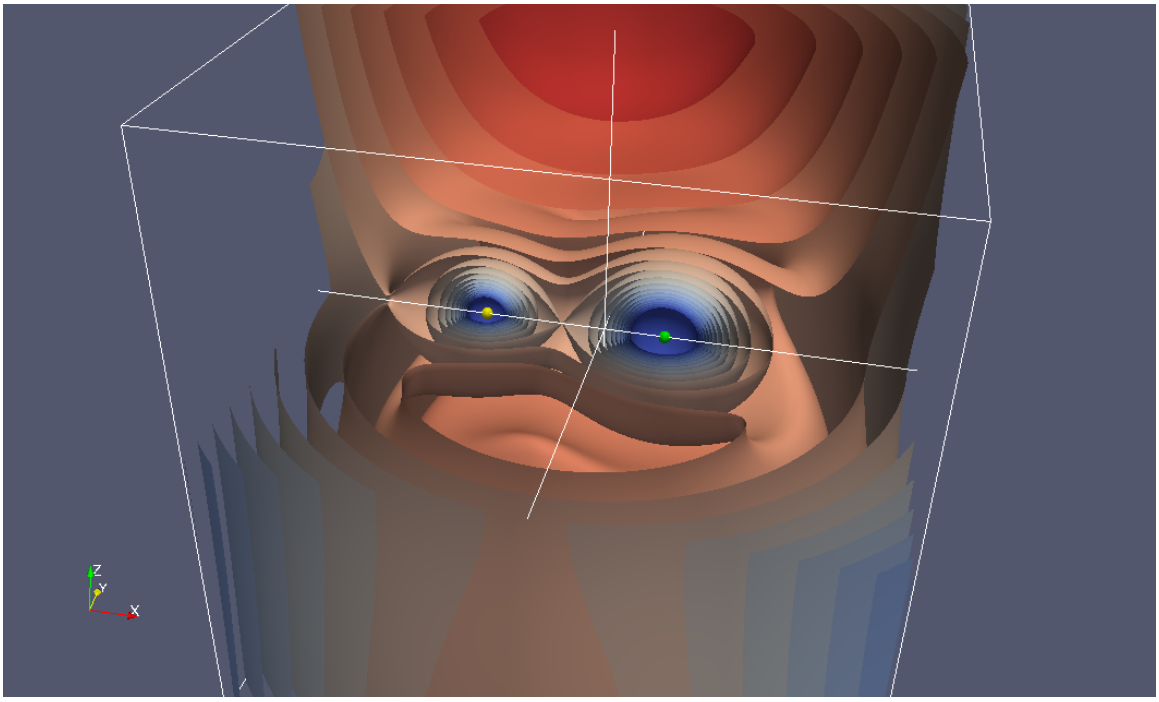

Figure 1: Roche lobes

The advantage of this equation is that, for circular orbits, the vectors $\mathbf{r}_\mathrm{p}$ and $\mathbf{r}_\mathrm{s}$ have fixed components with respect to the co-rotating frame. Comparing against the inertial equation (36) two new terms have appeared, corresponding to two 'fictitious' or 'inertial' forces which exist to correct for the fact that the co-rotating frame isn't inertial. The first, Coriolis term is non-zero only when the velocity $\dot{r}_\mathrm{c}$ is non-zero, whereas the second, centrifugal term is present whenever the particle is displaced from the rotation axis.

Equation (43) still doesn't easily yield close-form solutions, but is nevertheless quite instructive. In situations where the particle is at rest in the co-rotating frame, the net force on it is

$$\mathbf{F} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{p} - \mathbf{r}) + \frac{GM_\mathrm{s}m}{|\mathbf{r}_\mathrm{s} - \mathbf{r}|^3}(\mathbf{r}_\mathrm{s} - \mathbf{r}) \tag{44}$$

This force can be written as the gradient of an 'effective' potential,

$$\mathbf{F} = -\nabla\phi_\mathrm{eff}, \tag{45}$$

where

$$\phi_\mathrm{eff} \equiv \frac{m}{2}|\Omega \times \mathbf{r}|^2 - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|} - \frac{GM_\mathrm{p}m}{|\mathbf{r}_\mathrm{p} - \mathbf{r}|} \tag{46}$$

combines the gravitational potential from the two stars (the second and third terms on the right-hand side) with a centrifugal potential. At points where the gradient of the effective potential vanishes, the net force on the test particle will vanish, meaning that these are points of equilibrium.

Fig. 1 is a 3-D rendering of the effective potential for a system in which $M_\mathrm{p} = 2M_\mathrm{s}$. Close to each of the stars, the surfaces of constant ϕ_eff are nearly spherical, because one or the other of the gravitational terms in eqn. (46) dominates. Far from the rotation axis, the centrifugal term dominates and the equipotential surfaces are concentric cylinders. There are five points in total where $\nabla\phi_\mathrm{eff}$ vanishes, and these (four of which are visible in the figure) are the famous Lagrange points, traditionally labelled as $L1, \ldots, L5$. L1 through L3 lie on the line joining the two stars, with L1 between the stars, and L2 and L3 outside the stars (the convention being that L2 is closer to the smaller-mass star, and L3 to the larger-mass star). L4 and L5 lie in the orbital plane, such that they each form equilateral triangles with the two stars.

It is clear from the figure that L1, L2 and L3 are not local minima of $\phi_{\rm eff}$, but rather saddle points: the potential is *maximal* along the line joining the stars, and *minimal* along perpendicular lines. Thus, a particle at one of these points is not in stable equilibrium, and will drift away over time. How, then, do we speak of parking a satellite at one of the Lagrange points?[7] Although the points themselves are not stable, it is possible to place an object in a stable, periodic orbit near (not exactly on) of one of the points — a so-called 'halo' orbit. The orbit is maintained by the Coriolis force, which can't be modelled as an effective potential (and hence cannot be included in the definiton of $\phi_{\rm eff}$).

The L4 and L5 points are local maxima of $\phi_{\rm eff}$, suggesting that they, also, are unstable. Surprisingly, however, as a particle placed at one of these points begins to drift away, the Coriolis force will send it into a stable orbit about the point, *if* the mass ratio of the system exceeds about $25:1$. This underscores the fact that the effective potential does not tell the whole story about the motion of a test particle in the restricted three-body problem.

On a final note, as we shall see later in the course, the shapes of self-gravitating gasous bodies — i.e., stars — are defined by equipotential surfaces (in just the same way that sea level on Earth is an equipotential surface). In binary systems, these surfaces are known as Roche lobes, and correspond exactly to the surfaces shown in Fig. 1. Clearly, as the radius of one star grows (e.g., due to evolutionary effects), its surface becomes progressively distorted by the other star and the centrifugal force; until eventually the surface reaches out to the L1 point, and star will begin to spill mass onto the other star. This phenomenon is 'Roche lobe overflow', and we'll be returning to it later in the course.

[7] For instance, the plan is to park the *James Webb Space Telescope* at L2.