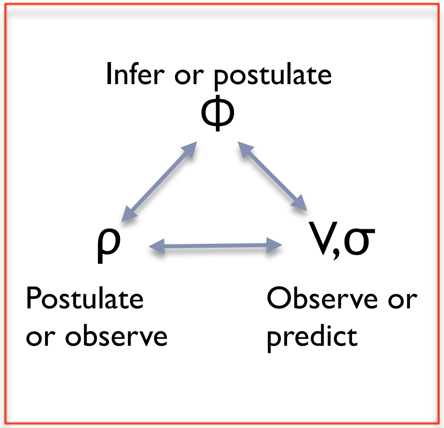
Astronomy 730

Dynamics

Outline

- Galactic dynamics
  - Potentials & energetics (3-25)
    - Characteristics of dynamical systems
  - Rotation curves and Tully-Fisher
    - Disk-halo degeneracy (26-40)
  - Dynamics of collisionless systems (41-70)
    - Collisionless Boltzman Equation
    - Disk mass, heating and stability

process



Infer or postulate

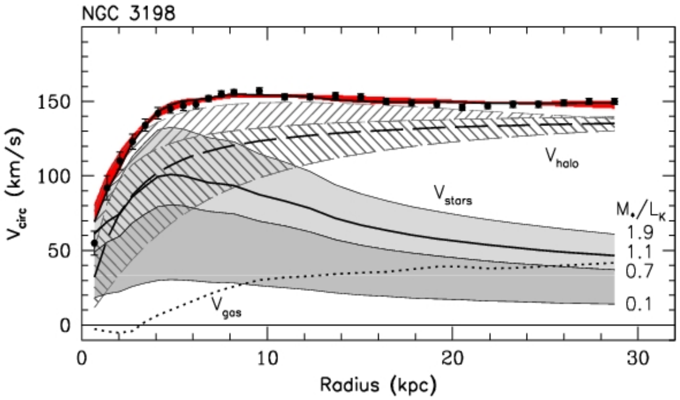
Φ

ρ

Postulate
or observe

V, σ

Observe or
predict



Galactic Dynamics

- Basic morphology of galaxies (and parts of galaxies) is determined by the orbits of stars
  - disk galaxies are disk-like because most of the stars orbit in nearly circular orbits in a flattened plane.
- What determines the stellar orbits? The gravitational potential: $\Phi(r,\theta,z)$.
- What determines the gravitational potential? The distribution of mass, ρ (r,θ,z).

Fundamentals: Gravitational Potentials

- Newton's gravitational force law for a point-mass M
  - $d(m\mathbf{v})/dt = -GmM\mathbf{r}/r^3$
    $= -m\blacktriangledown\Phi(\mathbf{r})$
    - $\mathbf{v},\mathbf{r}$ vectors; r scalar; $\blacktriangledown$ the gradient
    - Φ is the gravitational potential, $\Phi = -GM/r$
  - Thus, $F(\mathbf{x}) = -\blacktriangledown\Phi$
    - the force is determined by the gradient of the potential.

- Gravitational potential generalized:
  - $\Phi(\mathbf{x}) \equiv -G \int (\rho(\mathbf{x}')/|\mathbf{x}'-\mathbf{x}|)\, d^3\mathbf{x}'$
  - $F(\mathbf{x}) = G \int [(\mathbf{x}'-\mathbf{x})/|\mathbf{x}'-\mathbf{x}|^3]\ \rho(\mathbf{x}')d^3\mathbf{x}'$
    - Force on a unit mass at position, **x**, from a distribution of mass $\rho(\mathbf{x})$.
- Take the divergence of $F(\mathbf{x})$ $[\blacktriangledown \bullet F(\mathbf{x}) = -\blacktriangledown^2\Phi(\mathbf{x})]$ to get Poisson's equation:

$$\blacktriangledown^2\Phi(\mathbf{x}) = 4\pi G\,\rho(\mathbf{x})$$

- Directly related to Gauss's law:
  - In the absence of sources: $\blacktriangledown \bullet F(x) = 0$
  - Laplace equation: $\blacktriangledown^2\Phi(x) = 0$

See S&G and divergence theorem for derivation. Think:
What does "divergence" mean?

Fundamentals: Divergence theorem

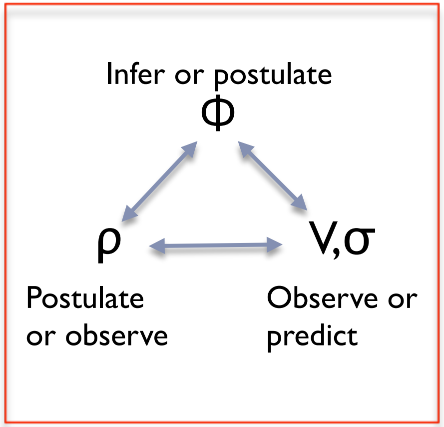
- Divergence theorem states that for some vector **F**
  - $\int \nabla \bullet \mathbf{F}\, dV = \int \mathbf{F} \bullet d\mathbf{S}$
    - Consider volume to be subdivided into a large number of small cells with volume ΔV_i.
    - For the cell-walls bounded by the surface, the sum of the surface-integrals for these cell-walls equals the surface-integral for the volume.
      - $\Sigma_i \int \mathbf{F} \bullet d\mathbf{S}_i = \int \mathbf{F} \bullet d\mathbf{S}$
    - For the remainder of surfaces, since the outward surface-normal of one cell is opposite that of the surface of the adjacent cell, the surface integrals cancel.
    - We can also write:
      - $\Sigma_i [(1/\Delta V_i) \int \mathbf{F} \bullet d\mathbf{S}_i]\ \Delta V_i = \int \mathbf{F} \bullet d\mathbf{S}$
    - In the limit where $\Delta V_i \rightarrow 0$, the sum of the surface-integrals becomes an integral over V
    - the ratio of the surface-integrals to ΔV_i as $\Delta V_i \rightarrow 0$ is the divergence of **F**.

Divergence theorem corollary

- For scalar and vector functions g and **F:**
  - $\int g\blacktriangledown\bullet\mathbf{F}\ dV = \int g\mathbf{F}\bullet d\mathbf{S} - \int (\mathbf{F}\bullet\blacktriangledown)g\ dV$

Application of potentials to galaxies

- Here's the process:
  - We start by looking at some very simple geometric cases
  - Define a few terms that help us think about and characterize the potentials
  - Become more sophisticated in the form of the potential to be more realistic in matching galaxies
- Concepts:
  - circular and escape velocities
  - Time scales: dynamical, free-fall
  - Potential (W or PE) and kinematic energy (K or KE)
  - Energy Conservation and Virial Theorem
  - Angular momentum
- Example: rotation curves of galaxies



Infer or postulate

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Energy considerations

- Recall:
  - $d(mv)/dt = - m\nabla\Phi(\mathbf{x})$,
- Take the scalar product with **v**
  - $\mathbf{v}\bullet d(m\mathbf{v})/dt + m\mathbf{v}\bullet\nabla\Phi(\mathbf{x}) = 0$
  - $\Rightarrow$ $d/dt\ [\ \tfrac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})\ ] \quad = 0$

    where $d\Phi(\mathbf{x}) / dt = \mathbf{v}\bullet\nabla\Phi(\mathbf{x})$
- Total energy defined:
  - $E = KE + PE = \tfrac{1}{2}\ mv^2 + m\Phi(\mathbf{x})$
- *This means E is constant for closed system*
  - e.g., an unperturbed orbit of a star
  - This is true for static potentials.
  - If there is a time varying potential (i.e. in a cluster) only the total energy is conserved (not the energy of an individual star)
  - For an external force add the summation of $\mathbf{F}_{ext}\cdot\mathbf{x}$

Kinetic energy and escape velocity

- If E is constant for closed system
- and by definition: KE ≥ 0
  - Implications:
    - As x ➔ ∞ (far from potential) $\Phi(x) \rightarrow 0$.
    - If $E > 0$ at $x=\infty$ then $v > 0$
    - i.e., the object has escaped the potential
  - Escape velocity for critical energy (E=0):
  - $\boxed{v_e(\mathbf{x}) = (2|\Phi(\mathbf{x})|)^{1/2}}$

Potential Energy

- Work (W) done in assembling a mass distribution is the potential energy.
- Start with initial portion of mass $\rho(\mathbf{x})$ which generates potential $\Phi(\mathbf{x})$
- Add an increment of mass δm: the work done is $\delta m \Phi(\mathbf{x})$
  - The work per unit mass over a distance $\mathbf{x}$ is
    - $\mathbf{F} \bullet \mathbf{x} = -\mathbf{x} \bullet \blacktriangledown \Phi(\mathbf{x})$
  - Integrating the work from x=∞ to some finite distance, where

    $\Phi(\mathbf{x} \rightarrow \infty) \rightarrow 0$

    implies $\Phi(\mathbf{x})$ as the total work (potential energy) per unit mass.
- Think of δm as equivalent to a change in density over the assembled volume:
  - $\int \delta \rho(\mathbf{x}) d^3\mathbf{x}$.
- Then work done, $\delta m \Phi(\mathbf{x})$ is:
  - $\delta W = \int \delta \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}$.

Potential Energy (continued)

- Apply Poisson's equation on $\delta\rho$ yields
  - $\delta W = (1/4\pi G)\int \Phi(\mathbf{x})\,\nabla^2(\delta\Phi)\,d^3\mathbf{x}$.
- Use the divergence theorem to write:
  - $\delta W = (1/4\pi G)\int \Phi(\mathbf{x})\,\nabla(\delta\Phi)\cdot d\mathbf{S} - (1/4\pi G)\int \nabla\Phi(\mathbf{x})\cdot\nabla(\delta\Phi)\,d^3\mathbf{x}$
- The surface-integral vanishes because:
  - $\Phi(\mathbf{r})$ and $|\nabla(\delta\Phi)|^{1/2}$ go as r^{-1} as $r \rightarrow \infty$
  - i.e, the integrand goes as r^{-3} while the surface area goes as r^2

✔

- Also there is this identify: $\nabla\Phi(\mathbf{x})\cdot\nabla(\delta\Phi) = \frac{1}{2}\,\delta|(\nabla\Phi)|^2$
- From which it follows:
  - $\delta W = -(1/8\pi G)\;\delta\left[\int|\nabla\Phi|^2 d^3\mathbf{x}\right]$

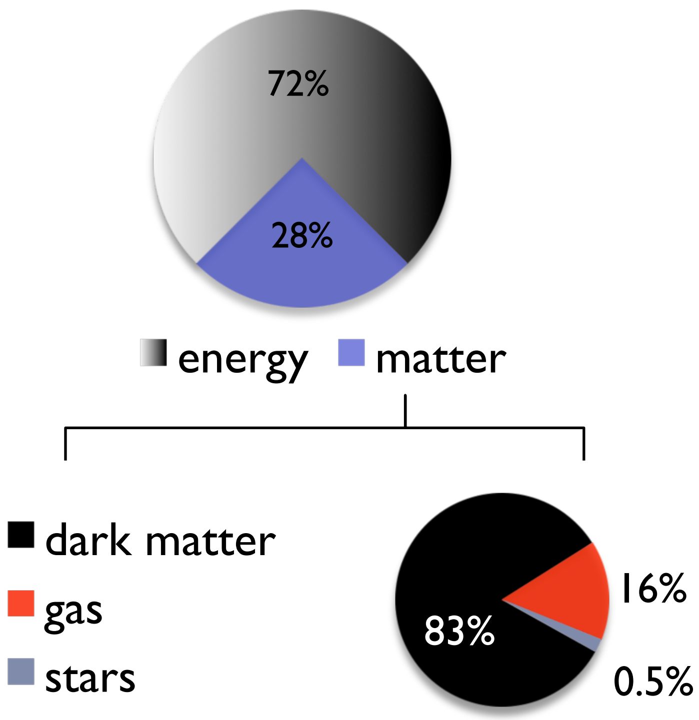
Why is this ok?

- Sum over all δW to get
  - $W = -(1/8\pi G)\int|\nabla\Phi|^2 d^3\mathbf{x}$
- Again apply Divergence theorem and Poisson equation to arrive at
  - $\boxed{W = \frac{1}{2}\int \rho(\mathbf{x})\;\Phi(\mathbf{x})\;d^3\mathbf{x}}$

Why we might care about potential energy

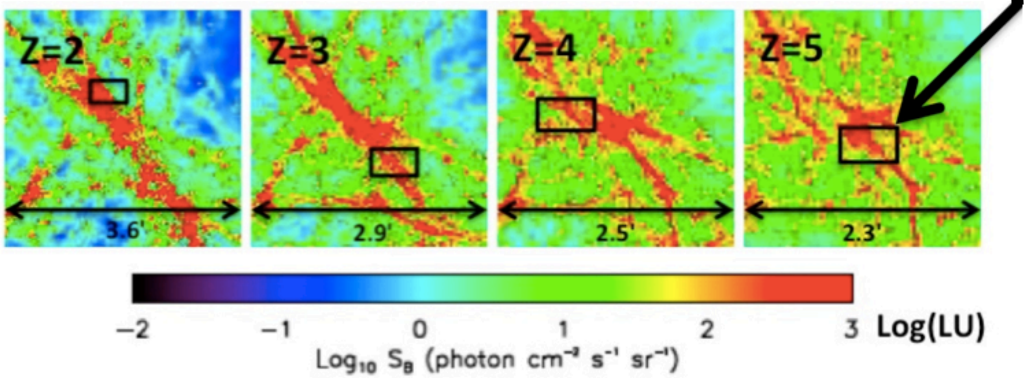
Where are the baryons?

WMAP (e.g., Bennett et al. '13)



We lucky few…

- ~5% of the energy-density
- ~17% of the mass in the universe
- Only ~3% of the baryons are bound in galaxies; the rest are presumably a very hot gas (cosmic web and halos)
  - Stellar Mass Fraction ~ 0.5% - *not well known!*



Ly α and OVI doublet (103.2, 103.8 nm; primary coolant for T=3e5K near solar gas, Otte+'03)

Do this problem:

- A galaxy forms, starting at t = t_{form}, out of matter initially at rest and dispersed over very large separations, $r_{form} \rightarrow \infty$. At t_{form}, the matter begins to collapse into a bound structure in dynamical equilibrium with total mass M_g and circular speed v_c. The galaxy virializes on a time-scale comparable to a few × t_{dyn}. In terms of these parameters:

a. What is the total energy, kinetic energy, and potential energy of the system before collapse occurs?

b. What is the total energy, kinetic energy, and potential energy of the system after it virializes? How do you account for differences in the total energy?

c. Estimate the characteristic size of the galaxy, and state what assumptions you have made to form this estimate.

d. Convert any total energy differences you find into the equivalent mass of hydrogen (in units of M_g) that can be ionized and heated to 10^6 deg (K) assuming all of the "missing energy" goes into this heating (hint: take a look at Problem 7.2 in Sparke & Gallagher). Also compute the numerical fraction of M_g for v_c = 200 km s^{-1}.

Adopt v_c =200 km s^{-1} for the next three questions:

e. Assuming the material out of which the galaxy forms is representative of the universe on average (see the lectures), what fraction of the galaxy's baryons does your answer in (c) represent?

f. Assuming the galaxy is depleted of a fraction of its baryons, such that its baryon mass-fraction is 10% today, what is the temperature of the expelled gas?

g. What might these calculations tell us about the origin of the warm-hot ionized medium (WHIM) in the cosmic web and the fraction of the total baryons in it?

Angular Momentum,Torque & Integrals of Motion

- $L \equiv \mathbf{x} \times m\mathbf{v}$
- $N \equiv$ torque $= \mathbf{x} \times \mathbf{F} = dL/dt = -m\mathbf{x} \times \nabla \Phi$
  - Torques is rate of change of angular momentum

- For a spherically symmetric galaxy, L is constant. For an axisymmetric galaxy (most galaxies), only the component parallel to the symmetry axis is constant
- Integrals of motion:
  - Any function of phase-space coordinates ($\mathbf{x}$,$\mathbf{v}$) that is constant along an orbit
  - In static potentional $\Phi(\mathbf{x})$:
    - $E(\mathbf{x},\mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})$ is an integral of motion
  - If $\Phi(R,z,t)$ is axisymmetric about z-axis:
    - L_z is an integral of motion
  - In spherical potential $\Phi(R,t)$:
    - all three components of L are integrals of motion.

Virial Theorem

- In an isolated system composed of multiple mass units, these masses can change (and exchange) their kinetic and potential energy as long as
  - the sum is constant (E is conserved overall):
    - <KE>+<PE> = constant
      - ☐ where <> are averages over the system
- If the system is in equilibrium, the kinetic energy must balance the potential energy, e.g., an object in a circular orbit around a fixed mass.
- This equilibrium configuration is referred to as "virialization", and implies:
  - $\boxed{2\langle KE\rangle + \langle PE\rangle = 0}$
- For an external force add the summation of $\mathbf{F}_{ext} \bullet \mathbf{x}$:
  - $2\langle KE\rangle + \langle PE\rangle + \mathbf{F}_{ext} \cdot \mathbf{x} = 0$

Virial Theorem in practice

- This is a primary tool for inferring masses of dynamical systems in astronomy, so it is important. Here's how it is applied:
  - Measure the velocity dispersion of some spherical system
    - (e.g., a star cluster).
  - We observe V_r, from which we can derive: $\sigma_r = <V_r>^{1/2}$
  - If we assume motions are isotropic, then $\mathbf{V} \bullet \mathbf{V} = \Sigma_i V_i^2 \sim 3\,\sigma_r^2$.
  - Kinetic energy = $(3/2)\ M\sigma_r^2$
  - Potential energy looks something like: $-GM^2/2fr_c$
    - r_c is a "core radius" at which point the surface brightness is ½ of its central value.
    - f is a fudge-factor or order unity that accounts for the details of the mass distribution in the integral $\int \rho(\mathbf{x})\ \Phi(\mathbf{x})\ d^3\mathbf{x}$.
  - Solve for mass, M

Recap: Energy considerations

- Recall:
  - Newton: $d(mv)/dt = - m \nabla \Phi(\mathbf{x})$,
  - Poisson: $\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x})$
  - Total energy defined: $E = KE + PE = \frac{1}{2} mv^2 + m\Phi(\mathbf{x})$
  - $PE = W = \frac{1}{2} \int \rho(\mathbf{x}) \, \Phi(\mathbf{x}) \, d^3\mathbf{x}$
  - Escape velocity (E=0): $v_e(\mathbf{x}) = (2|\Phi(\mathbf{x})|)^{1/2}$
- Virial Theorem: $KE = -\frac{1}{2} PE$ for systems in equilibrium
  - In practice: this is the way we measure masses for astronomical objects because KE measured independent of potential and PE is proportional to M.

Recap: Virial Theorem

- Why is KE = −½ PE for equilibrium?
- Define moment of inertia
  - $I = m\ \mathbf{x}\bullet\mathbf{x}$
- Then it follows:
  - $\frac{1}{2}\ d^2(I)/dt^2 = m\ d/dt\ (\ \mathbf{x}\bullet d\mathbf{x}/dt\ )$
  - $= m\ d/dt\ (\mathbf{x}\bullet\mathbf{v})$
  - $= m\ [\ d\mathbf{x}/dt\bullet\mathbf{v} + \mathbf{x} \bullet d\mathbf{v}/dt\ ]$
  - $= m\ v\bullet v + d/dt(m\mathbf{v})\bullet\mathbf{x}$

$\boxed{2\ KE = mv^2}$ $\boxed{-m\blacktriangledown\Phi(\mathbf{x})\bullet\mathbf{x} = PE}$

- For a system in equilibrium: $d^2(I)/dt^2 = 0$
  - I is time-independent
  - Is dI/dt interesting?

Spherical mass distributions

- *Start simple….*
- Newton showed:
- A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.
  - Mass contained in solid-angle $\delta\Omega$ of shell as seen by body depends on distance to shell:
    - $\delta m = \Sigma \, \delta\Omega \times r^2$, where Σ is the mass–surface–density of the shell.
    - Hence in any two directions:
      - $\delta m_1 / \delta m_2 = (r_1/r_2)^2 \rightarrow \delta F_1 = - \delta F_2$
      - particle is attracted equally in opposite directions
    - $▼\Phi = -F = 0$
- The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.
  - $\Phi = -GM/R$

Spherical distributions: characteristic velocities

- The gravitational attraction of a density distribution, $\rho(r')$, on a particle at distance, r, is:
  - $F(r) = -(d\Phi/dr) = -GM(r)/r^2$
  - $M(r) = 4\pi\int \rho(r')r'^2dr'$
- Circular speed:
  - In any potential $d\Phi/dr$ is the radial acceleration
  - For a circular orbit, the acceleration is v^2/r
  - ➔ $\boxed{v_c^2 = r(d\Phi/dr) = GM(r)/r}$
    - outside a spherical mass distribution, v_c goes as $r^{1/2}$
      - ☐ Keplerian
- Escape speed: $v_e(r) = (2|\Phi(r)|)^{1/2} = [2\int GM(r)dr/r^2]^{1/2}$

Homogeneous Sphere: characteristic time-scales

- $M(r) = (4/3)\pi r^3 \rho$
  - ρ is constant
- For particle on circular orbit, $v_c = (4\pi G \rho /3)^{1/2} r$
  - rises linearly with r.
  - Check out the Galaxy's inner rotation curve.
  - What does this say about the bulge?
- Orbital period: $T = 2\pi r/v_c = (3\pi/G \rho)^{1/2}$
- Now release a point mass from rest at r:
  - $d^2r/dt^2 = -GM(r)/r^2 = -(4\pi G \rho /3)r$
  - Looks like the eqn of motion of a harmonic oscillator with frequency $= 2\pi/T$
  - Particle will reach r = 0 in 1/4 period (T/4), or
  - $\boxed{t_{dyn} \equiv (3\pi /16G \rho)^{1/2}}$

Isochrone Potential

- *Since nothing is really homogeneous…*
- $\Phi(r) = -GM/[b+(b^2+ r^2)^{1/2}]$
  - b is some constant to set the scale
  - $v_c^2(r) = GMr^2/[(b+a)^2a]$ ➔ $(GM/r)^{½}$ at large r
  - $a \equiv (b^2+r^2)^{½}$
- This simple potential has the advantage of having constant density at small r, falling to zero at large r
  - $\rho_0 = 3M / 16\pi Gb^3$
- Similar to the so-called Plummer model used by Plummer (1911) to fit the density distribution of globular clusters:
  - $\Phi(r) = -GM / (b^2+ r^2)^{1/2}$
  - $\rho(r) = (3M / 4\pi Gb^3)\ (1+r^2/b^2)^{-5/2}$

Singular Isothermal Sphere

- Physical motivation:
  - Hydrostatic equilibrium: pressure support balances gravitational potential
  - $dp/dr = (k_B\ T/m)\ d\rho/dr = -\rho\ GM(r)/r^2$
    - $\rho(r) = \sigma^2/2\pi G r^2$
      - where $\sigma^2 = k_B\ T/m$
- Singular at origin so define characteristic values:
  - $\rho' = \rho/\rho_0$
  - $r' = r/r_0$
  - $r_0 \equiv (9\sigma^2 / 4\pi G\rho_0)^{1/2}$
- $\Phi(r)$ is straight-forward to derive given our definitions:
  - $\Phi(r) = V_c^2\ \ln(r/r_0)$
  - $v_c = 4\pi\rho_0\ r_0^2$
- A special class of power-law potentials for α=2
  - $\rho(r) = \rho_0\ (r_0/r)^\alpha$
  - $M(r) = 4\pi\rho_0\ r_0^\alpha\ r^{(3-\alpha)}/\ (3-\alpha)$
  - $v_c^2(r) = 4\pi\rho_0\ r_0^\alpha\ r^{(2-\alpha)}/\ (3-\alpha)$

Look what happens to V(r) when α=2

Pseudo-Isothermal Sphere

- Physical motivation: avoid singularity at $r=0$, but stay close to functional form. Posit:
  - $\rho(r) = \rho_0[1 + (r/r_c)2]^{-1}$
- $\Phi(r)$ is straight-forward to derive given our definitions
- $V(r) \quad = \quad (4\pi G \rho_0 r_c^2 [1-(r_c/r)\arctan(r/r_c)])^{1/2}$
  - This gives a good match to most rotation curves within the optical portion of the disk.
  - But it does not give a good description of the light distribution of disks.

Characteristics of dynamical systems - 1

- Summary:
  - $v_c \equiv \sqrt{(r\, d\Phi/dr)} = \sqrt{(GM(r)/r)}$, circular velocity
  - $v_e \equiv (2|\Phi|)^{1/2}$, escape velocity
  - $t_{dyn} \equiv \sqrt{(3\pi/16G\rho)}$
  - $t_{ff} \equiv \sqrt{(1/G\rho)}$, free-fall time $\sim t_{dyn}$
  - $t_{cross} \equiv R/v$, use characteristic radius and velocity

Characteristics of dynamical systems - 2

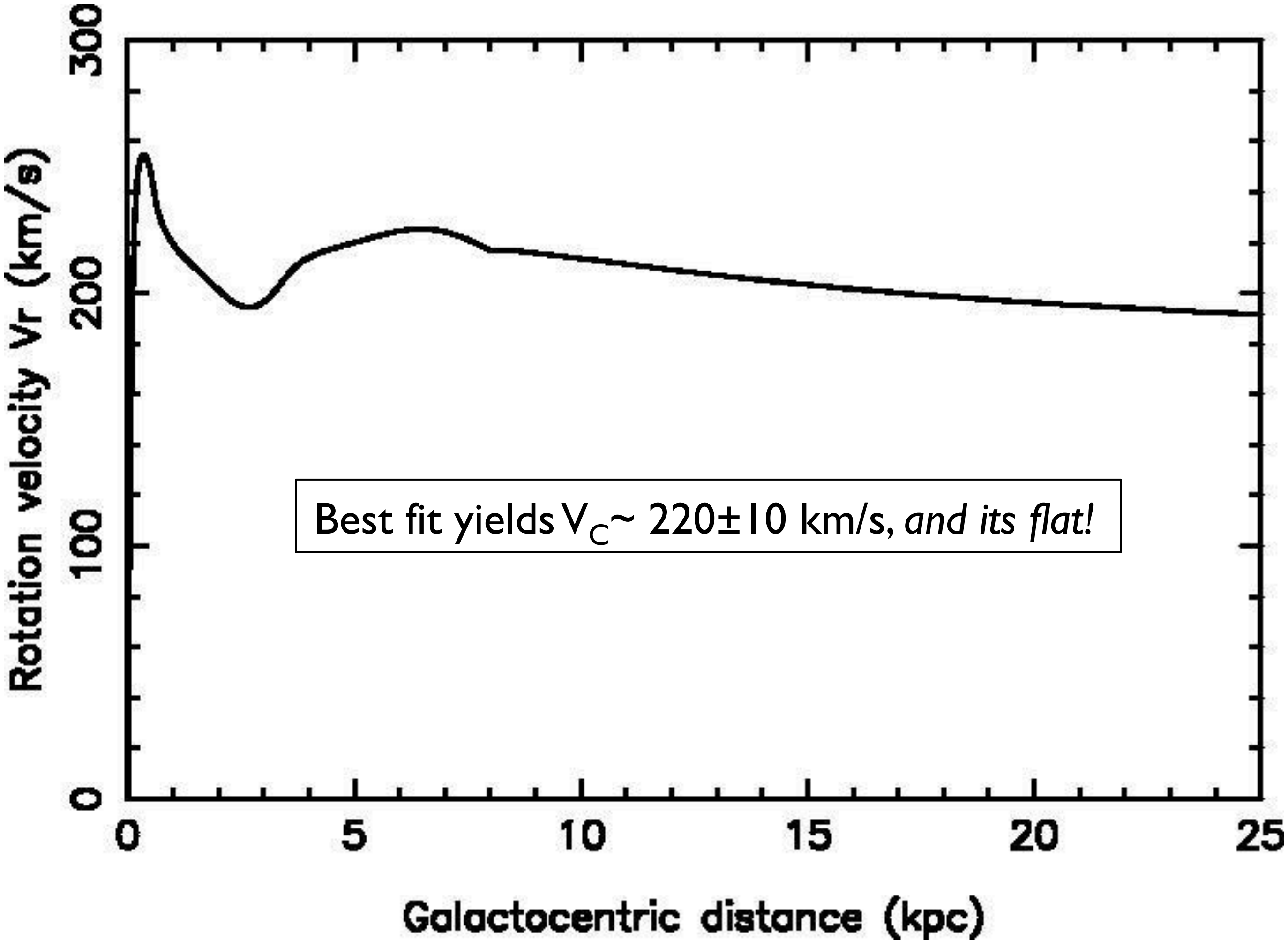
- Relaxation from N-body encounters of *stars*:
  - $t_s \equiv v^3 / (4\pi G^2 m_*^2 n)$, …time-scale for strong encounters
    - $\sim 4\times10^{12}$ yr $(v/10 \text{ km s}^{-1})^3 (m_*/M_\odot)^{-2} (n/1\text{pc}^{-3})^{-1}$
    - ➔ unimportant except in very dense star systems
  - However, many weak encounters cumulate such that after a time t_{relax}, the amplitude of the perturbed motion of the star is comparable to its initial motion:
  - $t_{relax} \equiv t_s / 2 \ln \Lambda$
    - $\sim 2\times10^{12}$ yr $(v/10 \text{ km s}^{-1})^3 (m_*/M_\odot)^{-2} (n/1 \text{ pc}^{-3})^{-1} (\ln \Lambda)^{-1}$
    - where $\Lambda = b_{max}/b_{min} \sim R/r_s = N/2$ for isolated system of N stars
      - ☐ when $\frac{1}{2} Nm_* v^2 \sim G(Nm_*)^2/2R$ and $r_s = 2Gm_*/V^2$
    - $t_{relax}/t_{cross} \sim N / 6 \ln N/2$
    - Still very large for realistic N (10^{10} to 10^{11} for galaxies)

Characteristics of dynamical systems - 3

- Instabilities to collapse: the Jean's length
  - $c_s \equiv \sqrt{}(k_B T / \mu\, m_H)$
    - sound-speed for temperature T and mol. mass $\mu\, m_H$
  - $\lambda_J \equiv cs\sqrt{}(\pi/G\rho) \sim c_s\, t_{ff}$
  - $M_J \equiv (\pi/6)\ \lambda_J^3 \rho\ =\ 20\ M_\odot(T/10K)^{3/2}(100cm^{-3}/n)^{1/2}$

  - What this basically says is that regions smaller than the sound-crossing time have time to re-arrange their density structure in response to gravity, and hence are stable *against* gravitational collapse; larger structures are unstable to collapse.
  - It is relevant for setting the mass-scales for star-formation and galaxy formation.

Flat rotation curves: the Milky Way



Best fit yields V_C~ 220±10 km/s, *and its flat!*

Nakanishi & Sofue (2003, PASJ, 55,191)

Flat rotation curves: external galaxies

Which looks most like the MW?
Why different shapes and extents?

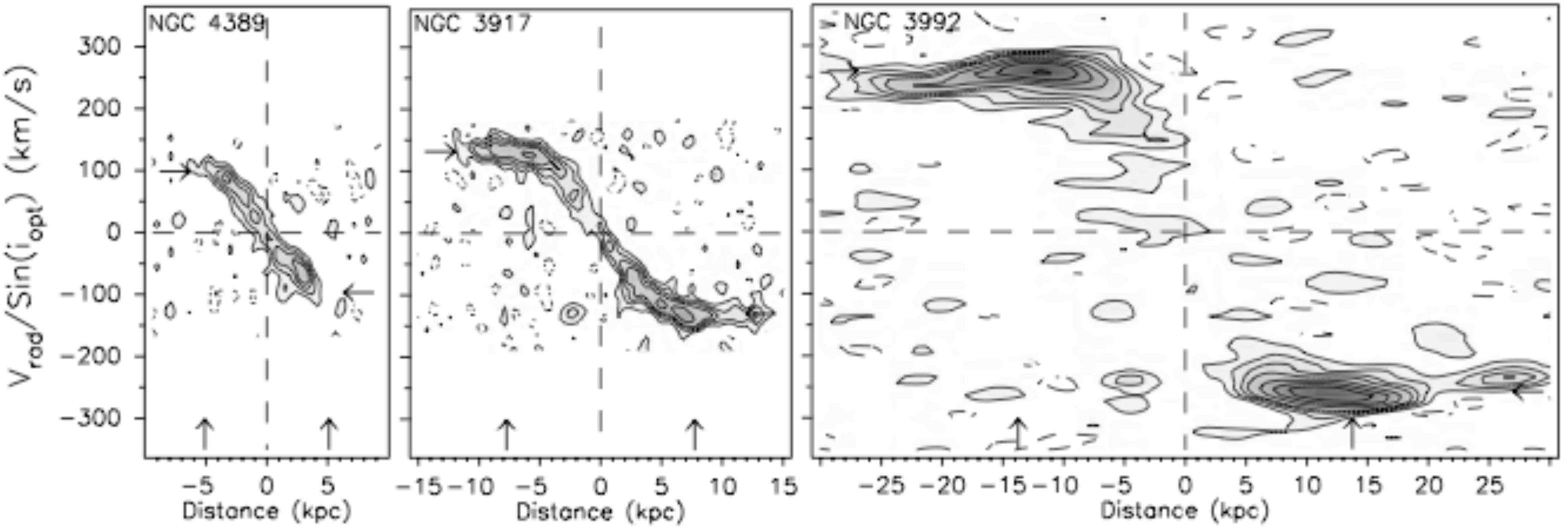


FIG. 2.—Prototype examples of the three categories of rotation curves. *Left:* Galaxy with a rotation curve that rises continuously until the last measured point. The measured maximum rotational velocity V_{max} is set by the extent of the H I disk (*R* curve). *Middle:* "Classical" rotation curve; a gentle rise in the central regions with a smooth transition into the extended flat part (*F* curve). *Right:* Rotation curve that reaches a maximum in the optical regions after which it declines somewhat to an extended flat part in the outer disk. In this case, the maximum rotation velocity exceeds the amplitude of the flat part (*D* curve). The vertical arrows indicate $\pm R_{25}$, and the horizontal arrows indicate the rotational velocities as inferred from the global profiles.

Verheijen 2005, ApJ, 563, 694

Flat rotation curves: external galaxies

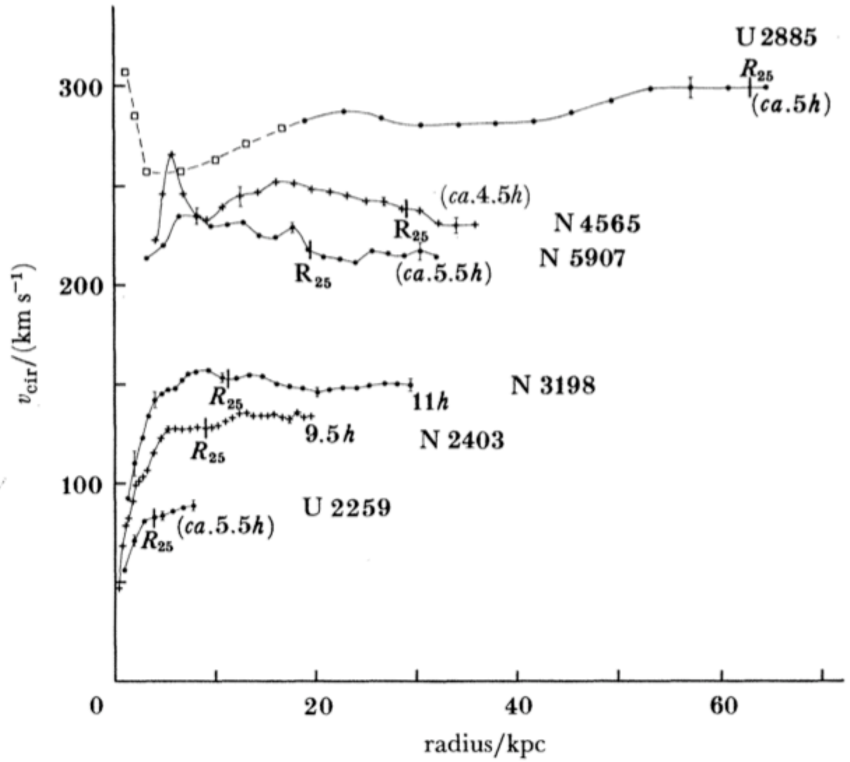


FIGURE 2. HI rotation curves for a number of spiral galaxies (Sancisi & van Albada 1986). Distances are based on $H_0 = 75$ km s^{-1} Mpc^{-1}. The optical radius, R_{25}, and the number of disc scalelengths, h, at the last measured point are indicated. For the inner region of UGC 2885 optical velocities (Rubin *et al.* 1986) have been used. All curves remain approximately flat beyond the turnover radius of the disc (2.5 h).

Van Albada et al. 1986, *Phil. Trans. Royal Soc. London*, 320, 1556, 447

Flat rotation curves: the disk

- Disk component
- $\Sigma(r) = \Upsilon \times \mu(r)$
  - Σ is the mass surface-density
  - Υ is the mass–to–light ratio (M/L)
  - μ is the surface–brightness
  - Surface mass density ($M_{\odot}$ pc^{-2}) is just the mass to light ratio times the surface brightness (L pc^{-2})
- Mass ➔ potential ➔ circular velocity
  - The trick here is to deal with the non-circular density distibution.

Flat rotation curves: the exponential disk

- $\Sigma(r) = \Sigma_0 \exp(-r/h_R)$
- Mass:
  - $M(r) = 2\pi\int \Sigma(r')r'\,dr' = 2\pi\,\Sigma_0 h_R^2[1-\exp(-r/h_R)(1+r/h_R)]$
- ➔ potential
  - $\Phi(r,z=0) = -\pi G\,\Sigma_0\, r\,[I_0(y)K_0(y) - I_1(y)K_1(y)]$
    - $y = r/2h_R$
    - I, K are modified Bessel functions of the 1st and 2nd kinds.

A bit of work; see Freeman (1970) and Toomre (1963)

- ➔ circular velocity
  - $V_c^2(r) = r\,d\Phi/dr = 4\pi G\,\Sigma_0\, h_R\, y^2\,[I_0(y)K_0(y) - I_1(y)K_1(y)]$

  - Note: This is for an infinitely-thin exponential disk. In reality, disks have a thickness with axis ratios $h_R{:}h_z$ between 5:1 and 10:1

Rotation from an exponential disk

Disk oblatness:
$q = h_z/h_r$

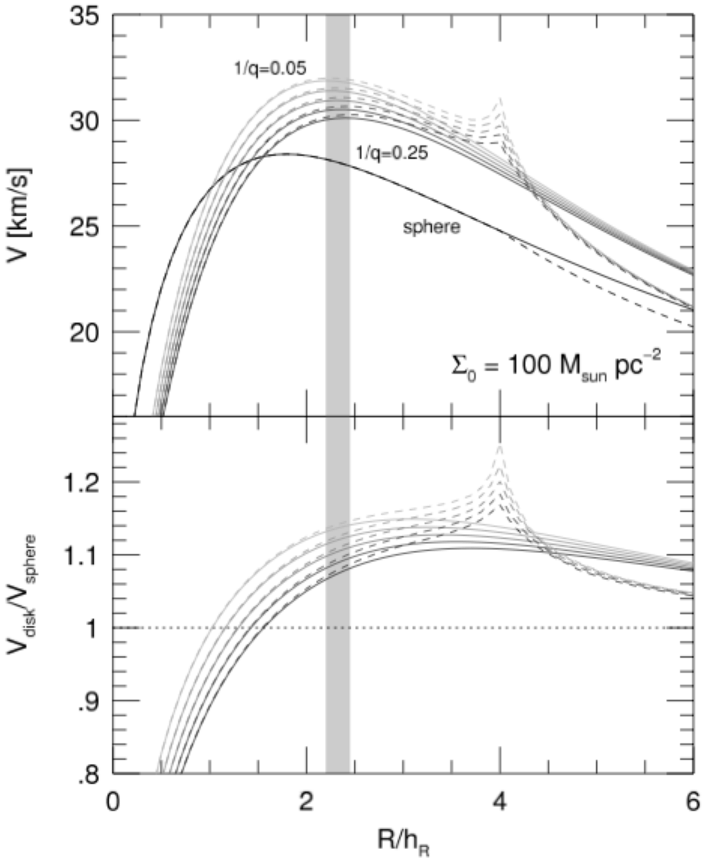
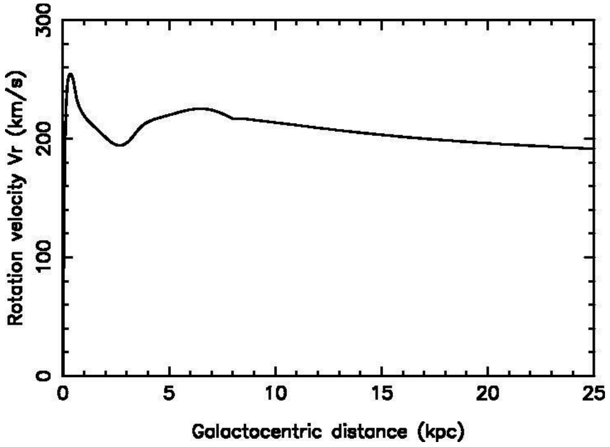


Fig. 17.— Rotation speed of an exponential disk with central mass surface density of 100 $\mathcal{M}_\odot$ pc^{-2} and oblateness $0.05 < q < 0.25$ versus radius normalized by scale-length, compared to a spherical density distribution with the same enclosed mass. Bottom panel shows the ratio of spherical to disk velocities. Dashed and solid lines show disks truncated at R/h_R=4 and 10, respectively. The radial range where these disks have peak velocities is shaded in gray.

← It isn't flat

↓ It is pretty flat



Flat rotation curves: the halo

- "Halo" component – we need V(r) to be constant at large radius (the bulge helps only at small r).
- One option is the singular isothermal sphere, here V(r) is constant at all radii.
  - Is that plausible given observed rotation curves (e.g, MW)?
- Another option: the pseudo-isothermal sphere
  - $\rho(r) = \rho_0[1 + (r/r_c)^2]^{-1}$
  - $V(r) = (4\pi G \rho_0 r_c^2 [1-(r_c/r)\arctan(r/r_c)])^{1/2}$
  - This gives a good match to most rotation curves within the optical portion of the disk.
- Also the NFW* profile, motivated by cold-dark-matter (CDM) structure-formation simulation (see S&G p.117):
  - $\rho_{NFW}(r) = \rho_n(r/a_n)^{-1}[1 + (r/a_n)]^{-2}$
  - $V_{NFW}(r) = (4\pi G \rho_n a_n^2 [\ln(1 + (r/a_n))/(r/a_n) - 1/(1 + (r/a_n))])^{1/2}$

---

*Navarro, Frenk & White 1996

The Disk-Halo Degeneracy

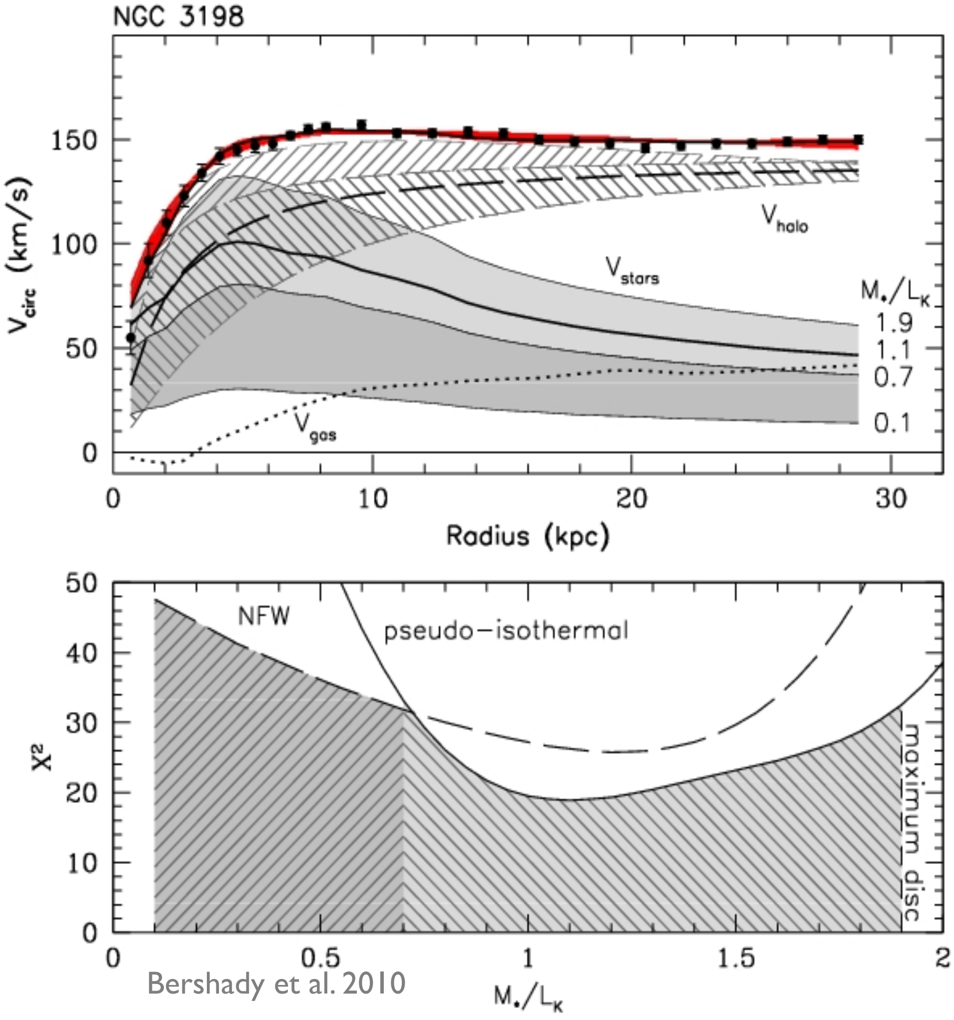
- **Q:** Is it possible to decompose the rotation curve of a spiral galaxy into disk, bulge, and halo components?
- Mass decomposition:
  - Recall $v_c^2 = r(d\Phi/dr)$
  - $\Phi = \Sigma_i \Phi_i$, i = bulge, disk, halo, kitchen sink
  - For spherical mass distribution
    - $r(d\Phi/dr) = GM(r)/r$
  - For a flattened mass distribution define f_i such that
    - $r(d\Phi/dr) = f_i GM(r)/r$
    - $v_c^2 = \Sigma_i f_i GM_i(r)/r = \Sigma_i v_{c,i}^2$
  - Measure v_c^2
  - Estimate individual components $v_{c,i}^2$ constrained by $v_c^2 = \Sigma_i v_{c,i}^2$
- Can it be done with any reasonable fidelity?

The Disk-Halo Degeneracy

- **Q:** Is it possible to decompose the rotation curve of a spiral galaxy into disk, bulge, and halo components?
- **A:** *No; Solutions are degenerate*
- Degeneracies:
  - Unconstrained fitting functions for halo:
    - e.g., pseudo-isotherm. vs NFW
  - Disk M/L (Υ_{disk}) uncertain
    - Stellar populations Υ_*: depends on SFH, IMF, and detailed knowledge of all phases of stellar evolution.
    - ISM
      - Gas
        - Atomic: straightforward to measure
        - Molecular: harder to measure
      - Dust: probably insignificant
    - Dark matter?
  - Non-circular motions
- However, it is possible to set upper-limits on the disk (so-called maximum disks)

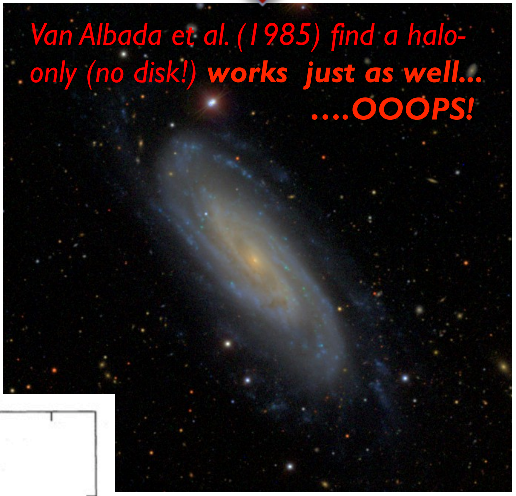
The Disk-Halo Degeneracy: Best case

- Rotation curve decomposition constraints:
  - Maximum disk - *yes*   Minimum disk - ***NO***



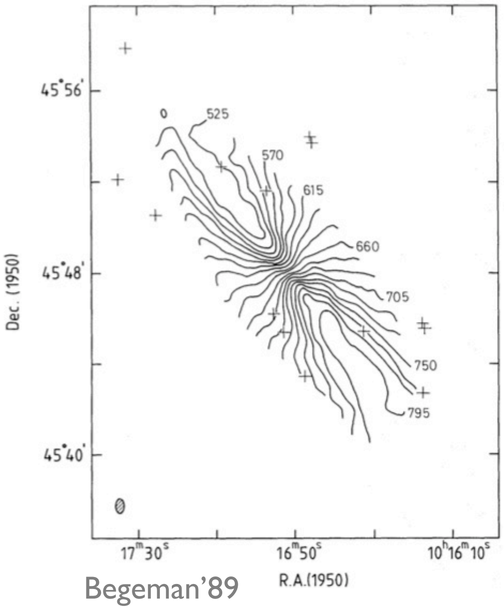
Bershady et al. 2010

← *Degenerate solutions…*



Van Albada et al. (1985) find a halo-only (no disk!) ***works just as well…***
….OOOPS!

SDSS gri



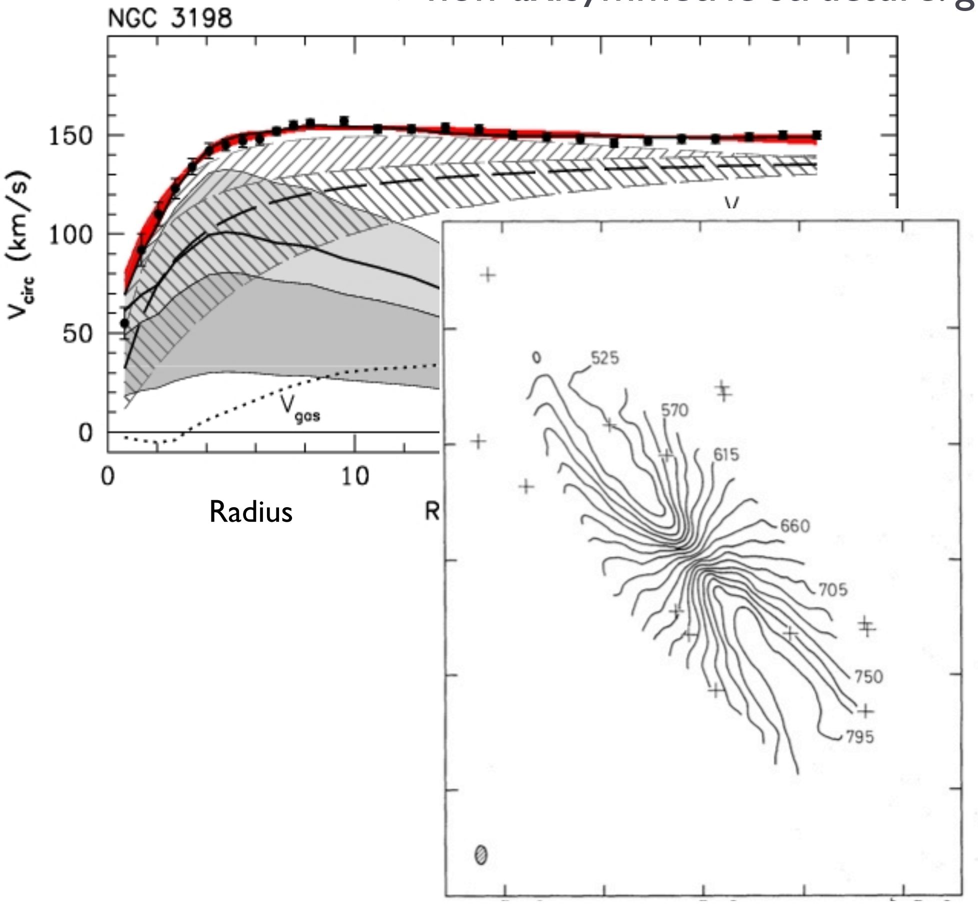
Begeman'89

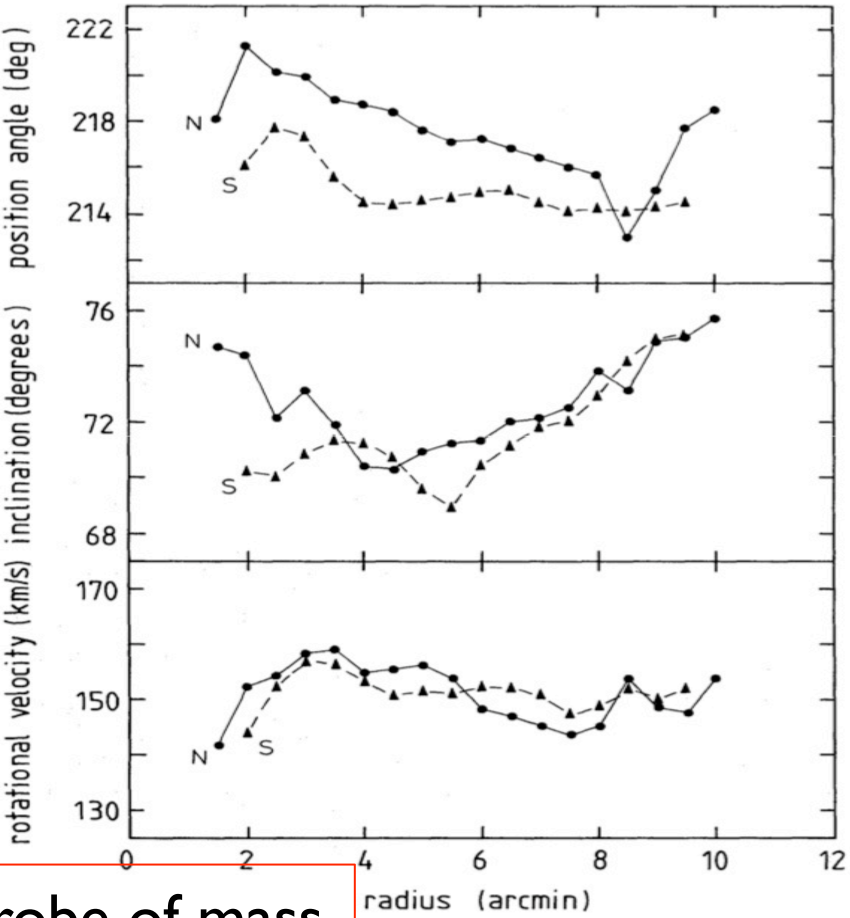
…it doesn't get better than this

←HI velocity field

The Disk-Halo Degeneracy: Best case

- Formal χ^2 not meaningful at level of ΔV_{circ}<5 km/s
  - non-axisymmetric structure: gas-flows, arms, winds, etc.





Begeman'89

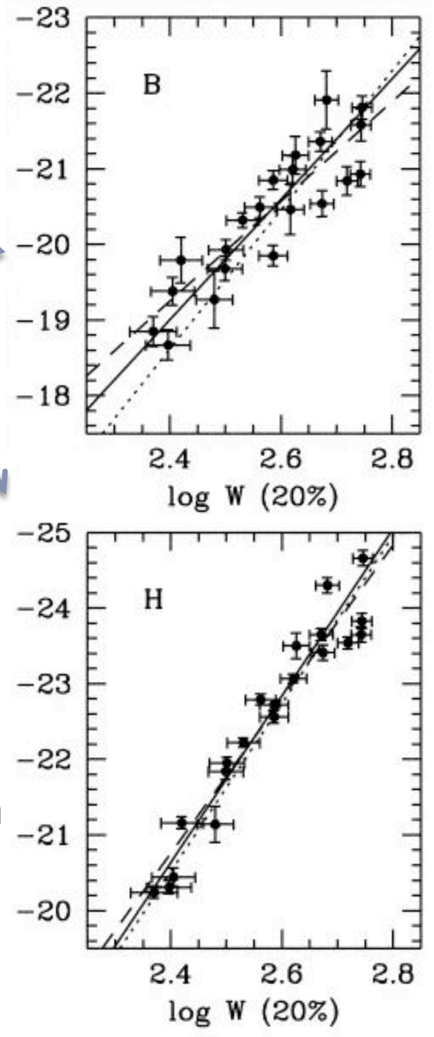
→ So we need another dynamical probe of mass

Tully-Fisher relationship: Scatter

- *Small!*
  - 0.5-0.3 mag in blue (B, 0.44 μ m)
  - 0.1 mag in near-IR (H, 1.6 μ m)
  - 0 mag (!) *intrinsic:* K-band for subset of galaxies with rotation curves and flat V(R) (Verheijen 2001)
    - *Too small?*

gasp!

Why this trend?

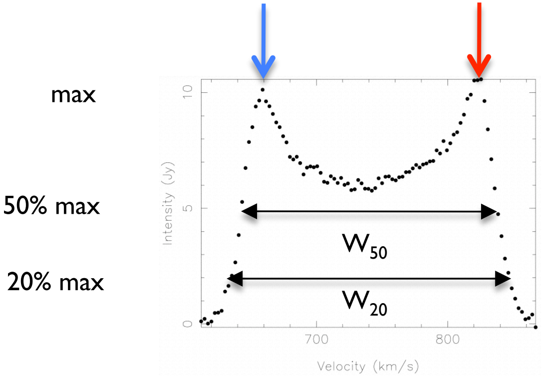


- Source of dispersion
  - Measurement errors (random)
  - Measurement errors (systematic)
    - Extinction
    - Shape of light-distribution (oblateness) → inclination
    - Shape of rotation curve → V_c
  - Cosmic variance
    - Variations in M/L with galaxy type

Surrogates measures of rotation

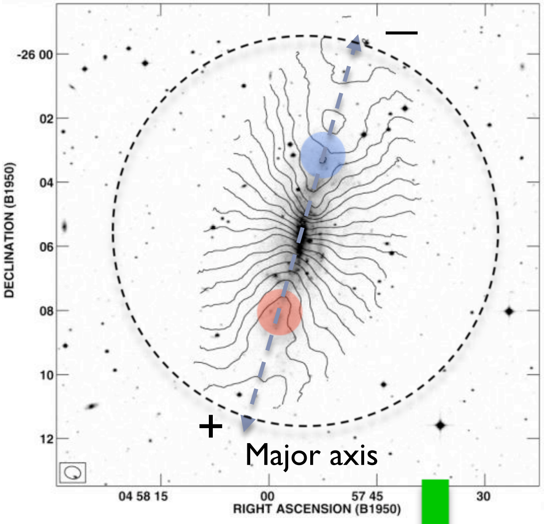
- Spatial information vs sensitivity:

4. Single dish (fiber):
Line width W ~ 2 V_c



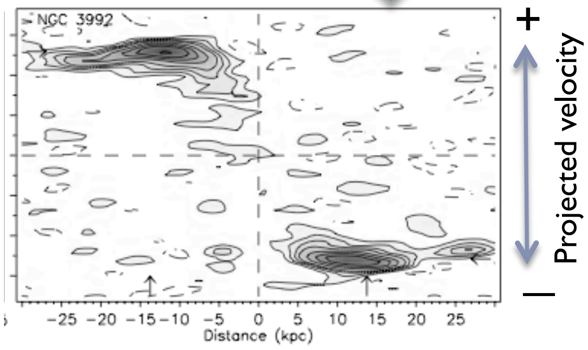
1. Interferometer/IFU:

➔Velocity field
2D map of velocities,
or data cube



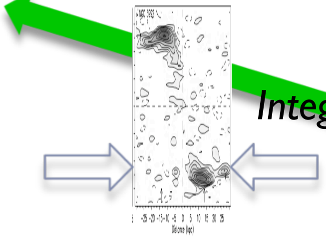
Slice down the major axis

2. Position-velocity diagram (PVD):
Equivalent to long-slit spectrum

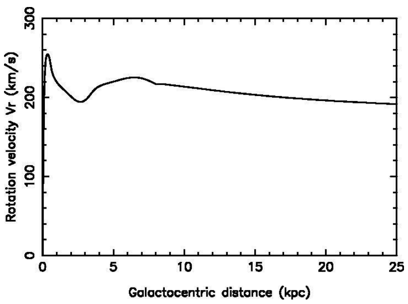


Major axis

3. Rotation curve



Integrate in x (and y)



Flip (in V) and fold (in x)

Tully-Fisher relation: Implications

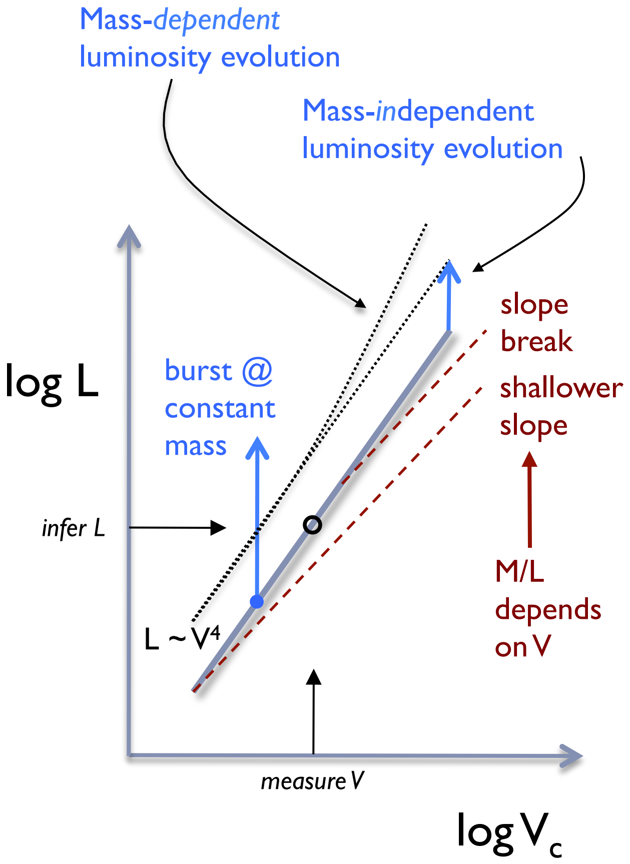
- Why is M/L so constant from galaxy to galaxy?
  - Here we're talking about M/L of the entire galaxy:
    - Mass is dominated by dark halo
    - Luminosity is dominated by disk
  - Total mass: M proportional to $[V^2_{max} h_R]$
  - Total luminosity: L proportional to $[I_0 h_R{}^2]$ (ignoring bulge)
  - ➔ L proportional to $[V^4_{max} / (M/L)^2 I_0]$
  - A universal M/L implies remarkable constancy of the ratio of dark to luminous matter
    - Or worse, a fine-tuning of the dark-to-luminous mass ratio as the stellar M/L varies.
- What does this tell us about galaxy formation and feedback?

Tully-Fisher relation: diagnostic tool

- Standard candle: V is distance-independent

- Structural probe: slope and scatter
  - Since L is proportional to [V^4_{max} / $(M/L)^2 I_0$]
  - ➔ *M* vs log(V) should have slope of 10
  - and should depend on surface-brightness
    - *Slope is < 10, varies with wavelength*
    - *No dependence on surface-brightness*

- Evolutionary probe
  - Changes in M/L with *time*
    - Assume M roughly constant
      - Secular changes in L: star-formation history
      - Stochastic changes in L (star-formation bursts)
        - Scatter increases with burst duty-cycle



Dynamics of collisionless systems

- Motivation:
  - Circular rotation is too simple and v_c gives us too little information to constrain Φ and hence ρ (e.g., rotation curves)
  - Without Φ and hence ρ we can't understand how mass has assembled and stars have formed
    - We can't even predict how the Tully-Fisher relation should evolve
  - Gas is messy because it requires understanding hydrodynamics, and likely magneto-hydrodynamics.
  - At are disposal are stars, nearly collisionless tracers of Φ!

Dynamics of collisionless systems

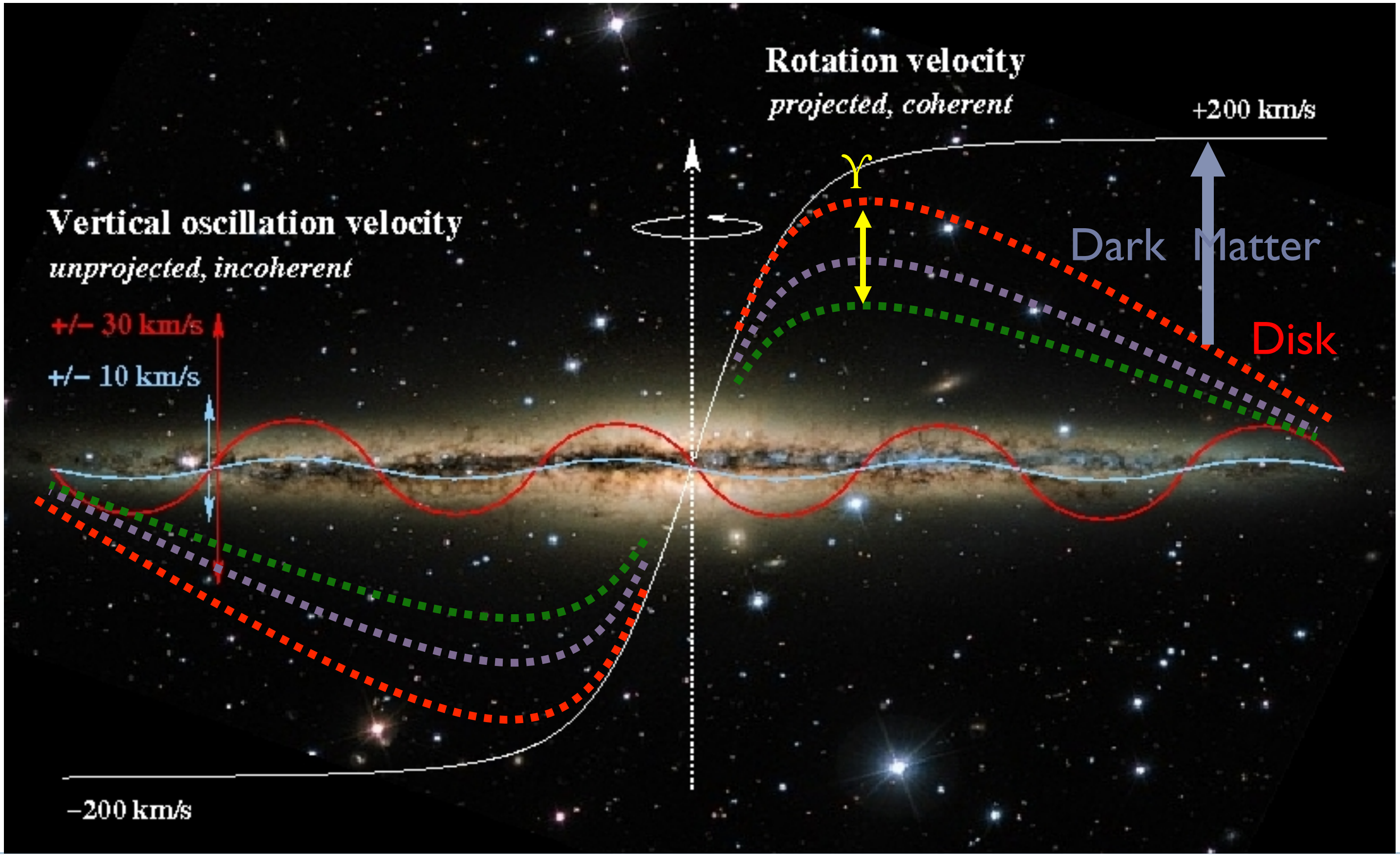
- How we'll proceed:
  - Start with the Continuity Equation (CE)
  - Use CE to motivate the Collissionless Bolztmann Equation (CBE), like CE but with a force term (remember $\blacktriangledown \Phi(\mathbf{x})$!)
  - Develop moments of CBE to relate v and σ and higher-order moments of velocity to Φ and ρ .
- Applications to realistic systems and real problems
  - Velocity ellipsoid
  - Asymmetric drift

Don't be intimidated by moment-integrals of differential equations in cylindrical coordinates: follow the terms, and look for physical intuition.

Disk heating
Disk mass
Disk stability

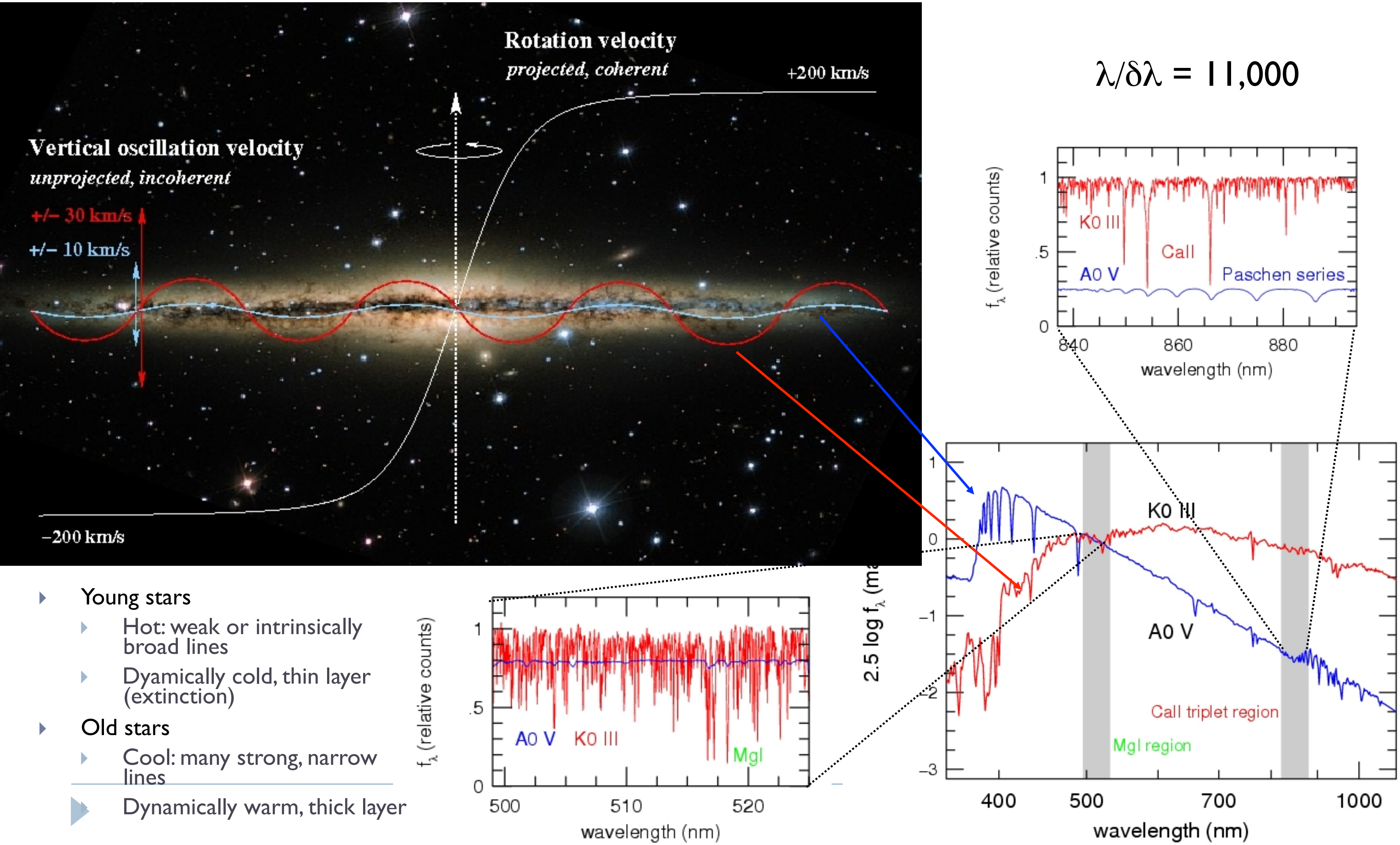
Example: Breaking the Disk-Halo Degeneracy

- Rotation provides the *total* mass within a given radius.
- Vertical oscillations of disk stars provides *disk* mass within given height



Vertical oscillations: a direct, dynamical approach

The kinematic signal



$\lambda/\delta\lambda = 11{,}000$

- Young stars
  - Hot: weak or intrinsically broad lines
  - Dyamically cold, thin layer (extinction)
- Old stars
  - Cool: many strong, narrow lines
  - Dynamically warm, thick layer

Disk Mass formula

Use *statistical* measure of *disk thickness* from edge-on galaxies …

vertical distribution*

thickness

vertical oscillations

$$\Sigma = 100\left(\frac{k}{3/2}\right)^{-1}\left(\frac{h_z}{444\text{ pc}}\right)^{-1}\left(\frac{\sigma_z}{30\,\text{km/s}}\right)^{2} \text{M}_{\text{sol}}\text{ pc}^{-2}$$

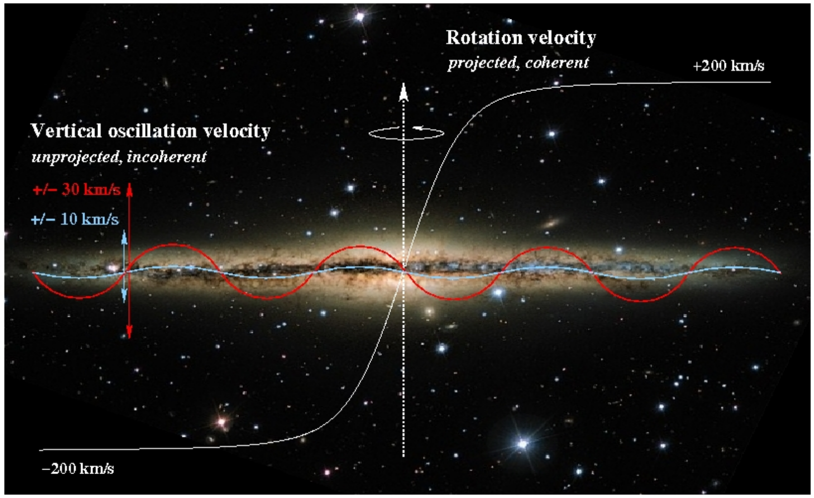
Disk mass surface density

….and apply relation to face-on galaxies where the *vertical oscillations* of stars can be measured.

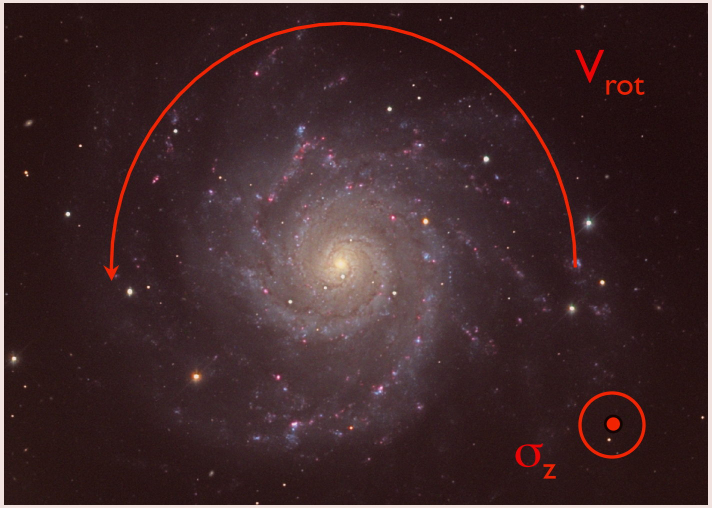
* $1.5<k<2$ for exp, sech, sech^2

Edge-on or Nearly Face-on ?

- ✓ Rotation projected
- ✗ Vertical dispersion *inaccessible* except via statistical *kinematic* correlations
- ✓ Vertical height projected



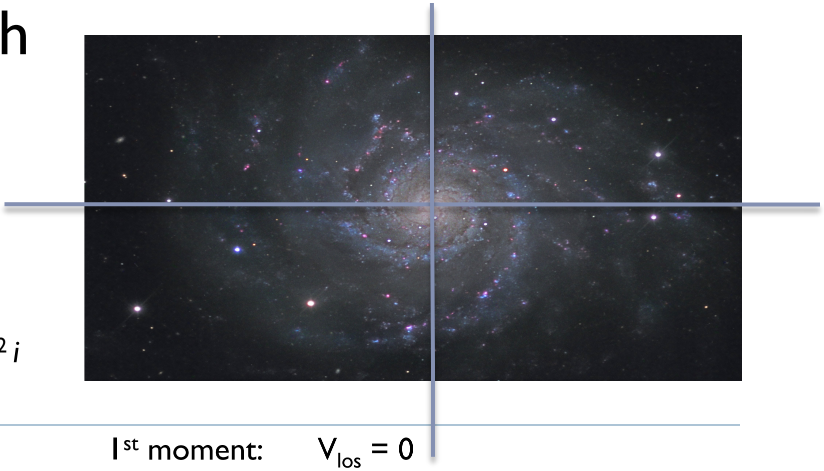
- ✓ Rotation accessible at high spectral resolution
- ✓ Vertical dispersion projected
- ✗ Vertical height *inaccessible* except via statistical *photometric* correlations



The problem

- If you look at completely face-on galaxies you can't measure rotation ➔ can't estimate total mass (total potential)
- Even if you look at *moderate inclination* (i~30°) galaxies, you get components of the stellar velocity dispersion (σ) which are not vertical (σ_z) but radial (σ_R) or tangential (σ_ϕ).
- In other words, $\boldsymbol{\sigma}$ is a vector – the velocity ellipsoid
- From the solar neighborhood we expect: $\sigma_R > \sigma_\phi > \sigma_z$
- But we can only observe 2 spatial dimensions:
  - How do we solve for σ_z ?
- And how do we solve for σ_R, which turns out to be interesting for understanding disk heating?

↓



1st moment: $V_{los} = V \sin i$

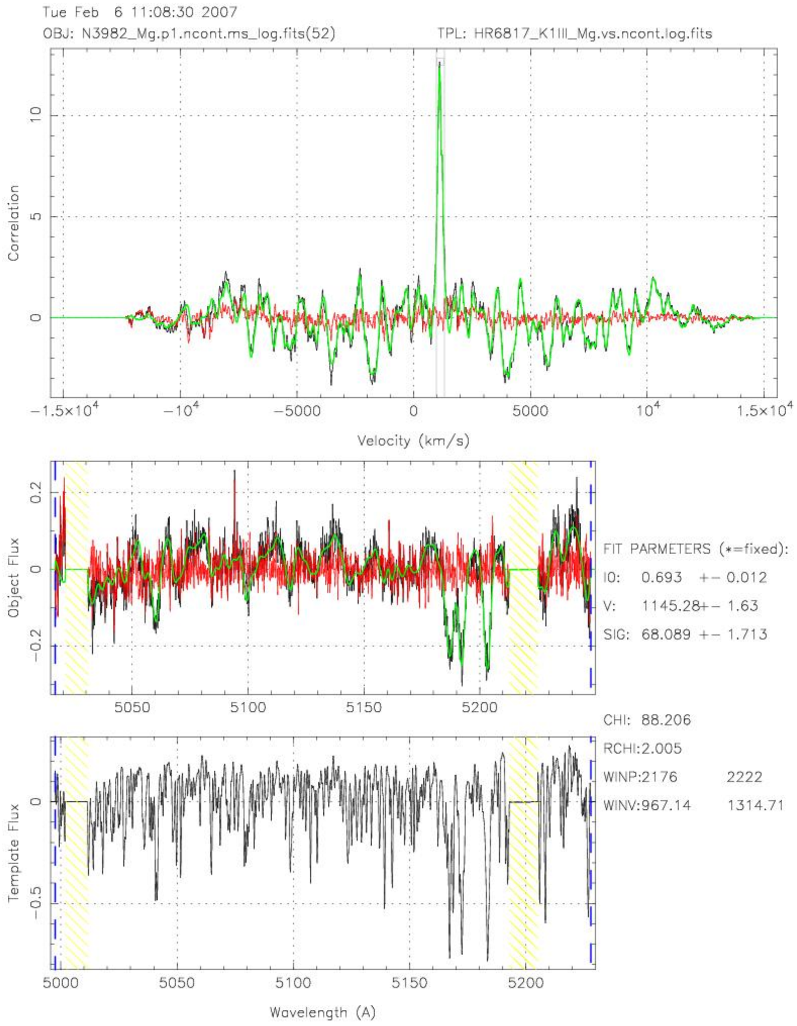
2nd moment: $\sigma_{los}^2 = \sigma_\phi^2 \sin^2 i + \sigma_z^2 \cos^2 i$

1st moment: $V_{los} = 0$

2nd moment: $\sigma_{los}^2 = \sigma_R^2 \sin^2 i + \sigma_z^2 \cos^2 i$

los = line of sight

Measuring the 2nd velocity moment



Tue Feb 6 11:08:30 2007

OBJ: N3982_Mg.p1.ncont.ms_log.fits(52)

TPL: HR6817_K1III_Mg.vs.ncont.log.fits

FIT PARMETERS (*=fixed):

I0: 0.693 +- 0.012

V: 1145.28+- 1.63

SIG: 68.089 +- 1.713

CHI: 88.206

RCHI:2.005

WINP:2176 2222

WINV:967.14 1314.71

Continuity Equation

- The mass of fluid in closed volume V, fixed in position and shape, bounded by surface S at time t
  - $M(t) = \int \rho(\mathbf{x},t) d^3\mathbf{x}$
- Mass changes with time as
  - $dM/dt = \int (d\rho / dt)\, d^3\mathbf{x} = -\int \rho\, \mathbf{v} \cdot d^2\mathbf{S}$
  - mass flowing out area-element d^2S per unit time is $\rho\, \mathbf{v} \cdot d^2\mathbf{S}$

NB: d = partial derivative

- The above equality allows us to write
  - $\int (d\rho / dt) d^3\mathbf{x} + \int \rho\, \mathbf{v} \cdot d^2\mathbf{S} = 0$
  - $\int [\, d\rho / dt + \nabla \cdot (\rho\, \mathbf{v})\,]\, d^3\mathbf{x} = 0$

Divergence theorem

- Since true for any volume
  - $\boxed{d\rho / dt + \nabla \cdot (\rho\, \mathbf{v}) = 0}$

This is CE

In words: the change in density over time (1st term) is a result of a net divergence in the flow of fluid (2nd term). Stars are a collisionless fluid.

Collisionless Boltzmann Equation

- Generalize concept of spatial density ρ to phase-space density $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$, **where** $f(\mathbf{x},\mathbf{v},t)$ is the distribution function (DF)
  - $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$ gives the number of stars at a given time in a small volume $d^3\mathbf{x}$ and velocities in the range $d^3\mathbf{v}$
  - The number-density of stars at location **x** is the integral of $f(\mathbf{x},\mathbf{v},t)$ over velocities:
    - $n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$ ⟷

$$\boxed{\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}}$$

  - The mean velocity of stars at location **x** is then given by
    - $\langle\mathbf{v}(\mathbf{x},t)\rangle = \int \mathbf{v}\, f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v} \,/ \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$

$$\boxed{\overline{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v}}$$

quantities you can measure

S&G notation

Notation we'll adopt

CBE *continued*

- **Goal**: Find equation such that given $f(\mathbf{x},\mathbf{v},t_0)$ we can calculate $f(\mathbf{x},\mathbf{v},t)$ at any t, …

  …. and hence our observable quantities $n(\mathbf{x},t)$, $<v(\mathbf{x},t)>$, etc.

  - $f(\mathbf{x},\mathbf{v},t_0)$ is our initial condition
  - The gravitational potential does work on $f(\mathbf{x},\mathbf{v},t)$

- Introduce some useful notation and relate to the potential
  - Let $\mathbf{w} \equiv (\mathbf{x},\mathbf{v}) = (w_1 \ldots w_6)$
  - $\mathbf{w}' \equiv dw / dt = (\mathbf{x}',\mathbf{v}') = (\mathbf{v}, -\nabla\Phi) = (w_1 \ldots w_3, -\nabla\Phi)$

CBE *continued*

- Recall CE gives: $d\rho/dt + \nabla\bullet(\rho\,\mathbf{v}) = 0$
- Replace $\rho(\mathbf{x},t) \rightarrow f(\mathbf{x},\mathbf{v},t)$
- CE gives:
  - $df/dt + \Sigma_{i=1,6}\, d(fw_i')/dw_i' = 0$ ….but:
    - $dv_i/dx_i = 0$ x_i,v_i independent elements of phase-space
    - $dv_i'/dv_i = 0$ $v' = -\nabla\Phi$, and the gradient in the potential does not depend on velocity.
  - $df/dt + \Sigma_{i=1,6}\, w_i'(df/dw_i) = 0$
  - $df/dt + \Sigma_{i=1,3}\,[v_i(df/dx_i) - (d\Phi/dx_i)(df/dx_i)] = 0$
- $$\boxed{df/dt + \mathbf{v}\bullet\nabla f - \nabla\Phi\bullet df/d\mathbf{v} = 0}$$

CBE

Vector notation

Getting something useful out of CBE

- CBE is the fundamental equation of stellar dynamics
- It is a special case of Liouville's theorem:
  - the flow of particles in phase space is incompressible, i.e.
  - phase-space density is constant.
- Unfortunately, general solutions to CBE are impractical.
- However, integral moments of the CBE and velocity provide useful *dynamical* relationships between components of the velocity vector, $\mathbf{v}$, the velocity ellipsoid, $\boldsymbol{\sigma}$, and the potential, Φ.
- This will look messy (it is), but very powerful results emerge.

CBE Integrals: warm up to learn tricks

- Start by integrating CBE over all velocities (0th moment)
- $\int \{ \ (df/dt)\, d^3\mathbf{v} + \Sigma_{i=1,3} [v_i(df/dx_i) - (d\Phi / dx_i)(df/dv_i)] = 0 \ \}$
  - We adopt summation convention
    - ☐ **A•B** = $\Sigma_{i=1,3} A_iB_i$ ➔ = A_iB_i ,
    - ☐ *i.e., repeated indices are implicitly summed over*

We assume the potential Φ is independent of velocity v_i

- $\int (df/dt)\, d^3\mathbf{v} + \int v_i(df/dx_i)\, d^3\mathbf{v} - (d\Phi / dx_i) \int (df/dv_i)\, d^3\mathbf{v} = 0$

range of velocities does not depend on time so d/dt comes outside integral and…

v_i range does not depend on x_i so df/dx_i comes outside integral and…

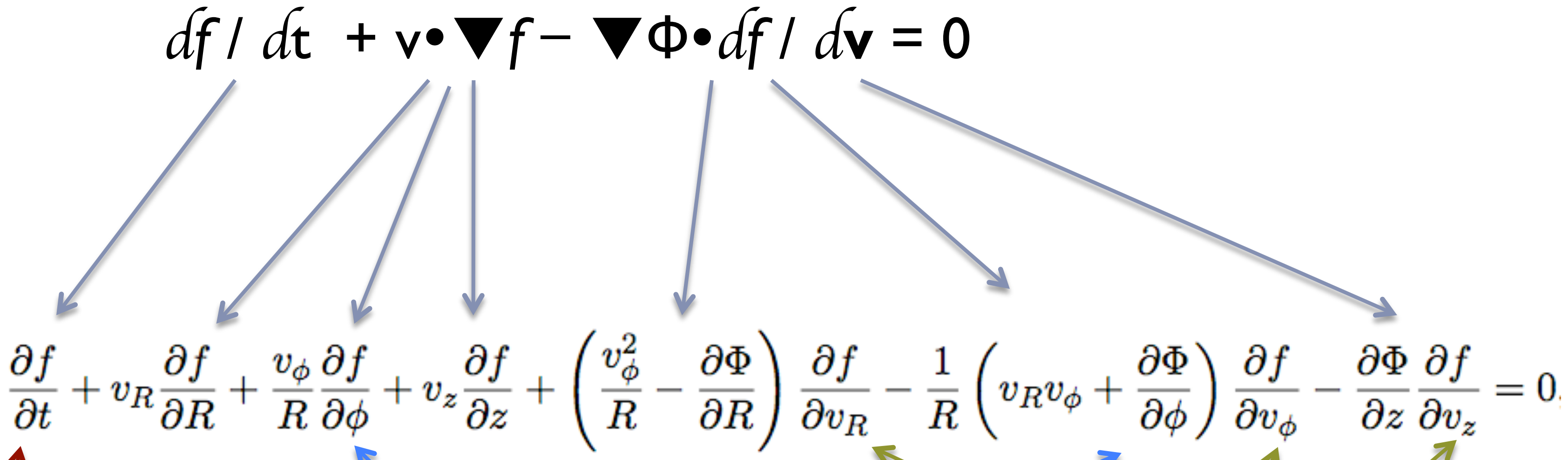
*Apply divergence theorem and the fact that f(**x**,**v**,t)=0 for sufficiently large |**v**|, i.e., at the surface of |**v**|➔∞*

Recall: $\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}$ and $\bar{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v}$

0

- $d\nu / dt + d(\nu\bar{v}_i)/ dx_i = 0$ ← *this is the continuity equation!*

Next: CBE in cylindrical coordinates



$$df\,/\,dt\ + \mathbf{v}\bullet\blacktriangledown f - \blacktriangledown\Phi\bullet df\,/\,d\mathbf{v} = 0$$

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0.$$

In what follows:

(1) The disk is in steady-state, so we can drop the first term

(2) we will assume the galaxy is azimuthally symmetric (e.g., a nice, circular, smooth disk) we can ignore all derivatives w.r.t. the azimuthal coordinate ϕ.

(3) The divergence theorem allows us to drop all integrals of velocity derivatives *unless* the moment is w.r.t. that velocity, in which case $v_i df/dv_i$ ➔ f, and: $\boxed{\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}}$

CBE-v_z moment: Surface-mass density Σ_{disk}

- Multiplying CBE by v_z , integrating over all velocities, assuming steady state, azimuthal symmetry, and using the divergence theorem yields:

$$\int v_z d^3\mathbf{v}\left\{ \cancelto{0}{\frac{\partial f}{\partial t}} + v_R\frac{\partial f}{\partial R} + \cancelto{0}{\frac{v_\phi}{R}\frac{\partial f}{\partial \phi}} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\cancelto{0}{\frac{\partial f}{\partial v_R}} - \frac{1}{R}\left(v_R v_\phi + \cancelto{0}{\frac{\partial \Phi}{\partial \phi}}\right)\cancelto{0}{\frac{\partial f}{\partial v_\phi}} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \right\}$$

$$\frac{\partial(\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{\nu\overline{v_R v_z}}{R} + \nu\frac{\partial \Phi}{\partial z} = 0$$

CBE-v_z moment: Σ_{disk} *continued*

- 1st and 3rd terms are smaller than 2nd and 4th by factors of $(z/R)^2$, and can be dropped.

$$\overset{x}{\frac{\partial(\nu\overline{v_R v_z})}{\partial R}} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \overset{x}{\frac{\nu\overline{v_R v_z}}{R}} + \nu\frac{\partial\Phi}{\partial z} = 0$$

- We also substitute the definition

$$\sigma_i^2 = \overline{v_i^2} - \overline{v_i}^2$$

  - Where $<v_i>$ (second term) is zero for a system in steady state

$$\boxed{\frac{\partial(\nu\sigma_z^2)}{\partial z} + \nu\frac{\partial\Phi}{\partial z} = 0}$$

(1) CBE

CBE-v_z moment: $\Sigma_{\rm disk}$ *continued*

- Now use Poisson's equation to define the potential Φ in cylindrical coordinates assuming azimuthal symmetry (no dependence of ν and Φ on ϕ):
  - $4\pi G\nu(\mathbf{x}) = \blacktriangledown^2\Phi(\mathbf{x}) = d^2\Phi/dz^2 + (1/R)\, d\,[R(d\Phi/dR)]\,/\,dR$
    - Remember: $\rho = \nu_i\, m_i = <\nu><m>$; we drop <> notation here
- For $d\Phi/dR = v^2(R)/R$ and $V(R)$ constant, the last term vanishes
- In general, in a highly flattened system near the mid-plane the 2nd term on the r.h.s. is much smaller than 1st term.

$$\boxed{\frac{\partial^2\Phi}{\partial z^2} = 4\pi G\ \nu}$$

(2) Poisson

CBE-v_z moment: Σ_{disk} *continued*

- Next, integrate Poisson over z and relate to CBE:

$$\int_{-z}^{+z} \frac{\partial^2\Phi}{\partial z^2}\, dz = 4\pi G \int_{-z}^{+z} \nu\, dz$$

Note $\frac{\partial\Phi}{\partial z} = 0$ at z = 0 by symmetry

$$= 2\,\frac{\partial\Phi}{\partial z}$$

← Indefinite integral
↓ Definite integral

$$\int_{-\infty}^{+\infty} \nu\, dz \equiv 4\pi G\, \Sigma_{\text{disk}}$$

Plug in to CBE

$$\boxed{\frac{\partial(\nu\sigma_z^2)}{\partial z} = -2\pi G\ \nu \int_{-z}^{+z} \nu\, dz}$$

(3) CBE+ Poisson

- To complete the calculation to find σ_z, integrate one more time in z.

CBE-v_z moment: Σ_{disk} *continued*

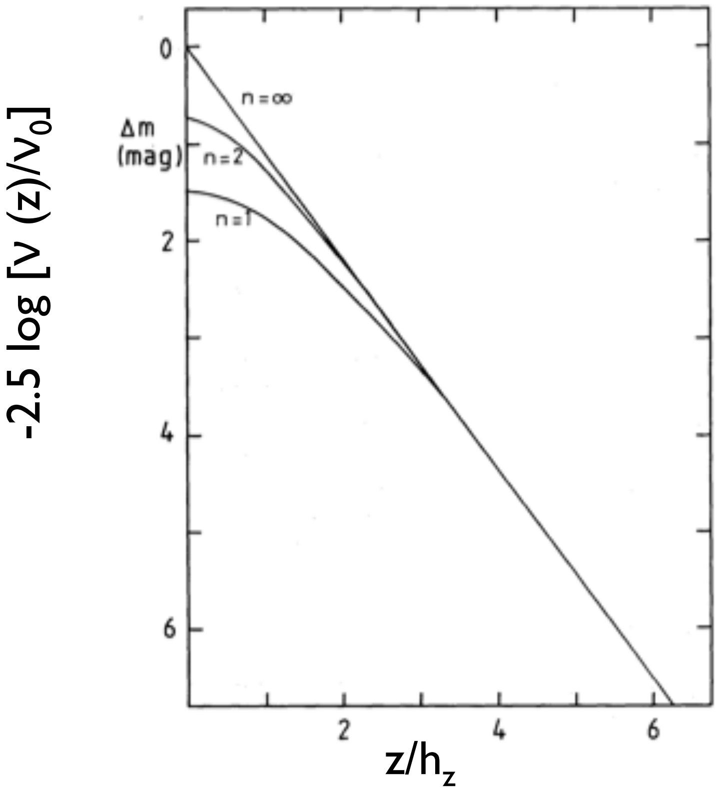
- To do this last step (integrate [3] in z), let's assume something about the mass distribution function in the vertical direction.
  - Based on what we know from light profiles of external galaxies:
    - $\nu(R,z) = \nu_0 \exp(-z/h_z - R/h_R)$
- Suggest a general vertical density function:
  - $\nu(z) = 2^{-2/n} \nu_0 \, \text{sech}^{2/n}(nz/2h_z)$
    - n=1 ➔ $\nu(z) = (\nu_0/4)\, \text{sech}^2 (z/2h_z)$ *isothermal case*
    - n=2 ➔ $\nu(z) = (\nu_0/2)\, \text{sech} (z/h_z)$ *intermediate*
    - n=∞ ➔ $\nu(z) = \nu_0 \exp(z/h_z)$ *what's observed (maybe)*
- The surface-density Σ_{disk} follows from direct integration:
    - n=1 ➔ $\Sigma_{\text{disk}} = \nu_0 h_z$
    - n=2 ➔ $\Sigma_{\text{disk}} = (\pi/2)\, \nu_0 h_z$
    - n=∞ ➔ $\Sigma_{\text{disk}} = 2\nu_0 h_z$

CBE-v_z moment: Σ_{disk} *continued*

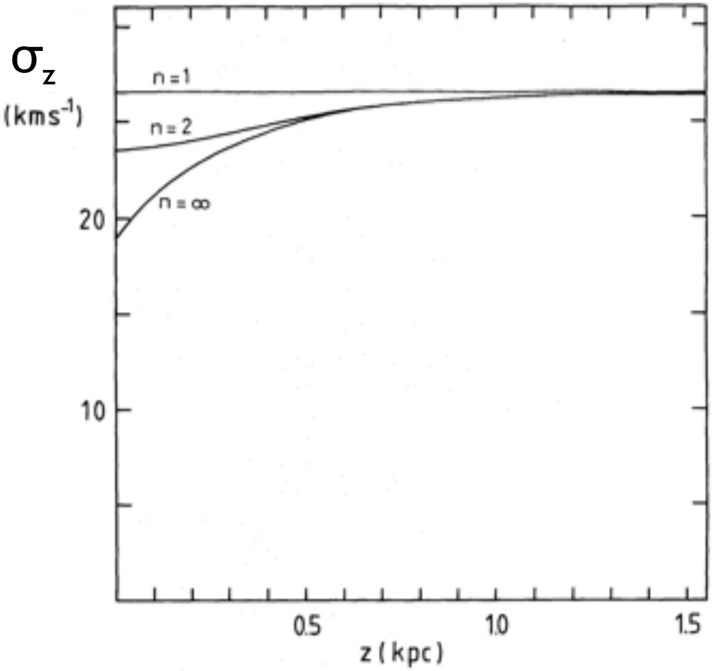
- The gradient of the potential follows from the corresponding indefinite integral:
  - $\frac{\partial \Phi}{\partial z} = 2\pi G \int \nu \, dz$
    - $= 2\pi G \nu_0 h_z \tanh(z/2h_z)$, n = 1
    - $= 2\pi G \nu_0 h_z \arctan[\sinh(z/h_z)]$, n = 2
    - $= 2\pi G \nu_0 h_z [1-\exp(z/h_z)]$, n = 3
- Lastly, we integrate the gradient of the potential $\frac{\partial \Phi}{\partial z}$ and divide by ν to solve for σ_z^2:
  - $\sigma_z^2 = 2\pi G \, h_z \, \Sigma_{\text{disk}}$ n = 1
  - $\sigma_z^2 = 1.7051 \, \pi \, G \, h_z \, \Sigma_{\text{disk}}$ n = 2
  - $\sigma_z^2 = 3\pi/2 \, G \, h_z \, \Sigma_{\text{disk}}$ n = 3

CBE-v_z moment: Σ_{disk} *continued*

- If the disk is locally isothermal, $d\sigma_z^2/dz = 0$
  - Why is this? What does isothermal mean in terms of kinematic motion?



$\Sigma_{\text{disk}} = 80\ M_\odot\ \text{pc}^{-2}$, h_z =325 pc



van der Kruit 1988, A&A, 192, 117

Finally….the Disk Mass formula

Use *statistical* measure of *disk thickness* from edge-on galaxies …

vertical distribution*

thickness

vertical oscillations

$$\Sigma = 100 \left(\frac{k}{3/2}\right)^{-1} \left(\frac{h_z}{444\ \mathrm{pc}}\right)^{-1} \left(\frac{\sigma_z}{30\,\mathrm{km/s}}\right)^{2} \mathrm{M_{sol}\ pc^{-2}}$$

Disk mass surface density

….and apply relation to face-on galaxies where the *vertical oscillations* of stars can be measured.

- * 1.5<k<2 for exp, sech, sech^2

CBE-v_R and $v_R v_\phi$ moments:

- Multiplying CBE by $v_R v_\phi$, integrating over all velocities, assuming steady state, azimuthal symmetry, and using the divergence theorem yields:

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z v_\phi})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v_\phi} R\frac{\partial\Phi}{\partial R} - 2\overline{v_R^2 v_\phi}\right) = 0$$

- Multiplying CBE by v_R, integrating over all velocities, and assuming azimuthal symmetry (ϕ-derivatives=0) yields:

$$\frac{\partial(\nu\overline{v_R})}{\partial t} + \frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial(\nu\overline{v_z v_R})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) = 0$$

CBE-v_R and $v_R v_\phi$ moments: Epicycle approximation

- The CBE-v_R and $v_R v_\phi$ moments combined with this identify (valid when ellipsoid is aligned with the potential andsymmetric about v_ϕ):

$$\overline{(v_\phi - \overline{v_\phi})^3} = (\overline{v_\phi^3} - \overline{v_\phi}\,\overline{v_\phi^2}) - 2\overline{v_\phi}(\overline{v_\phi^2} - \overline{v_\phi}^2) = 0$$

yield

$$\overline{v_R^2}\left(\frac{\partial\overline{v_\phi}}{\partial R} + \frac{\overline{v_\phi}}{R}\right) - \frac{2\overline{v_\phi}}{R}(\overline{v_\phi^2} - \overline{v_\phi}^2) = 0$$

- Which can be rearranged to give:

$$\boxed{\frac{\sigma_\phi^2}{\sigma_R^2} = \frac{1}{2}\left(\frac{\partial \ln\overline{v_\phi}}{\partial \ln R} + 1\right)}$$

This is powerful because it gives us another piece of information to uncover all of the ellipsoid components $\sigma_R : \sigma_\phi : \sigma_z$

CBE-v_R moment: Asymmetric drift

- Eliminating time derivatives and assuming there are no streaming motions ($<v_r>^2=0$) yields:

$$v_c^2 - \overline{v_\phi}^2 = \sigma_\phi^2 - \sigma_R^2 - \frac{R}{\nu}\frac{\partial(\nu\sigma_R^2)}{\partial R} - R\frac{\partial(\overline{v_r v_z})}{\partial z}$$

  - *Collisionless particles have tangential velocities smaller than the circular speed of the potential, in quadrature proportion (**think: energy**) to their velocity dispersion.*
  - This is powerful because it relates the velocity dispersion ellipsoid components to tangential velocities, thereby giving us another piece of information to uncover all of the ellipsoid components $\sigma_R : \sigma_\phi : \sigma_z$

- Now the problem is *over* constrained, i.e., σ_{maj}, σ_{min} plus *two* dynamical relations (epicycle approx. and asymmetric drift).
  - A *good* thing because there are a lot of assumptions.

Asymmetric drift

- Assume the gas tangential velocity is close to v_c
  - Why is this reasonable?
- V_ϕ is the tangential velocity of the stars

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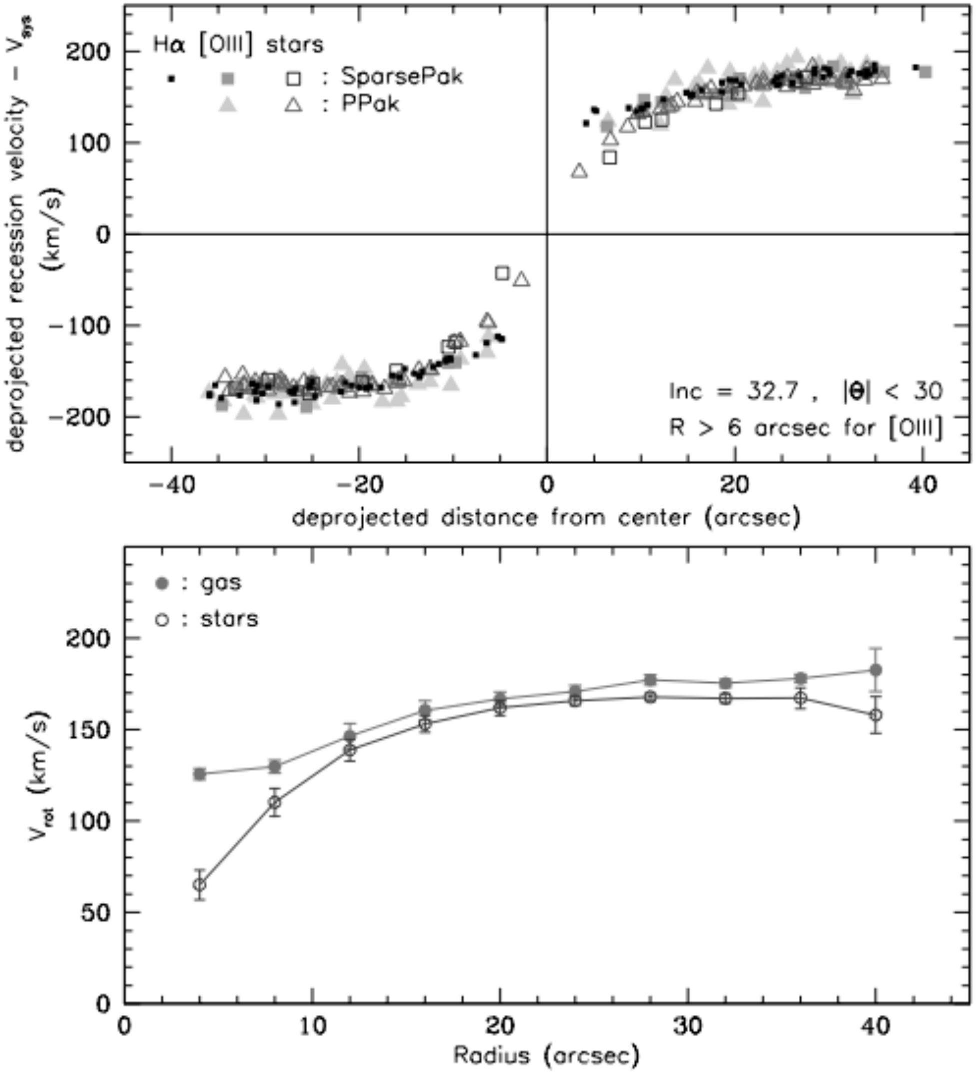


Bershady et al. 2010

Wrapping up:

- If we make some assumptions
  - about the distribution function $\nu(R,z)$, namely a double exponential in R and z,
  - that the ellipsoid tilt yields a last term between 0 and $\sigma_z{}^2$
  - and we substitute in the epicycle approximation to eliminate σ_ϕ

$$v_c^2 - \overline{v_\phi}^2 \approx \sigma_R^2 \left(\frac{1}{2} \frac{\partial \ln \overline{v_\phi}}{\partial \ln R} + \frac{2R}{h_\sigma} + \frac{R}{h_R} - 1 \right) + \frac{\sigma_z^2}{2}$$

- This formula, plus direct measurments of
    - v_c, $v_{\phi,}$ σ_{maj}, σ_{min}

  are our best-bet combination for
  - directly measuring Σ_{disk}
  - decomposing rotation curves,
  - determing disk M/L, and
  - the dark-mater density distribution.

Summary: CBE

- Continuity equation:
  - $d\rho / dt + \nabla \bullet (\rho \mathbf{v}) = 0$
- Collisionless Boltzmann equation
  - $df / dt + v \bullet \nabla f - \nabla \Phi \bullet df / d\mathbf{v} = 0$

Change in density over time (1st term) is a result of a net divergence in the flow of fluid (2nd term). Stars are a collisionless fluid.

Fundamental equation of stellar dynamics. Special case of Liouville's theorem: Flow of particles in phase space is incompressible, i.e. phase-space density is constant.

- General solution to CBE impractical but velocity-moments yield key relations between observables (v, σ) and the potential (Φ) for highly-flattened, axisymmetric systems:
  - v_z *moment*: **disk-mass surface density**

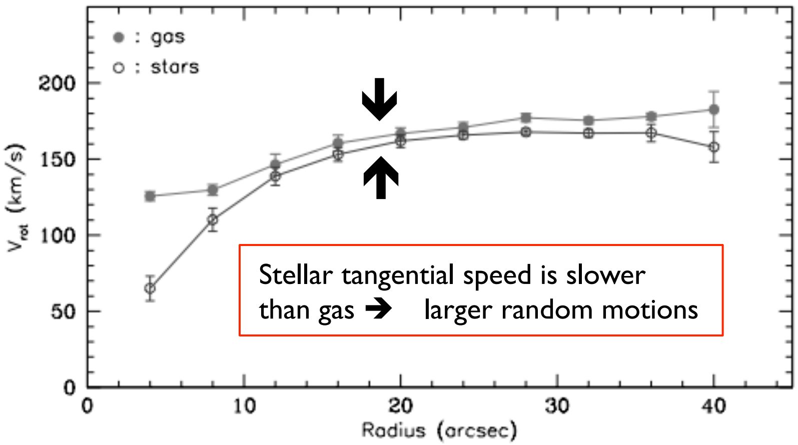
$$\sum = 100 \left(\frac{k}{3/2}\right)^{-1} \left(\frac{h_z}{444 \text{ pc}}\right)^{-1} \left(\frac{\sigma_z}{30 \text{ km/s}}\right)^{2} \text{M}_{\text{sol}} \text{ pc}^{-2}$$

  - v_R & $v_R v_\phi$ *moments*: **epicycle approx.**

$$\frac{\sigma_\phi^2}{\sigma_R^2} = \frac{1}{2}\left(\frac{\partial \ln \overline{v_\phi}}{\partial \ln R} + 1\right)$$

  - v_R *moment*: **asymmetric drift**

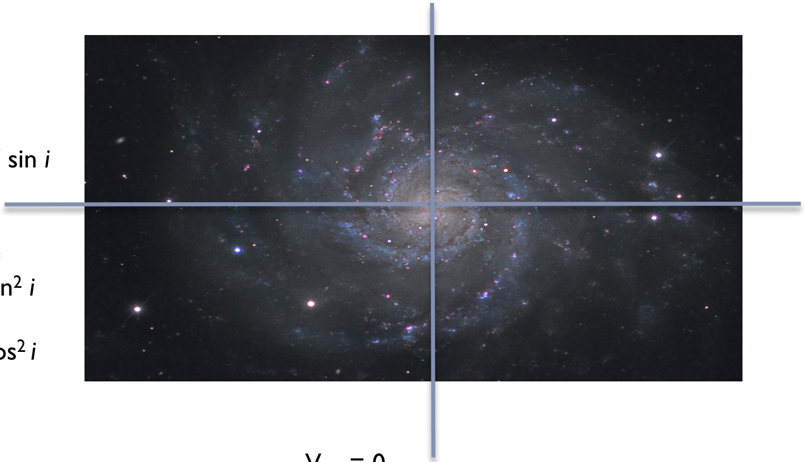
$$v_c^2 - \overline{v_\phi}^2 \approx \sigma_R^2 \left(\frac{1}{2}\frac{\partial \ln \overline{v_\phi}}{\partial \ln R} + \frac{2R}{h_\sigma} + \frac{R}{h_R} - 1\right) + \frac{\sigma_z^2}{2}$$



Stellar tangential speed is slower than gas ➔ larger random motions

Summary: CBE *applications*

- These formulae, plus direct measurements of
  - v_c, v_ϕ, σ_{maj}, σ_{min} *best-bet combination*
  - ➔ velocity ellipsoid $\boldsymbol{\sigma} = (\sigma_R, \sigma_\phi, \sigma_z)$
- directly measuring Σ_{disk}
- decomposing rotation curves,
- determing disk M/L
- the dark-mater density distribution

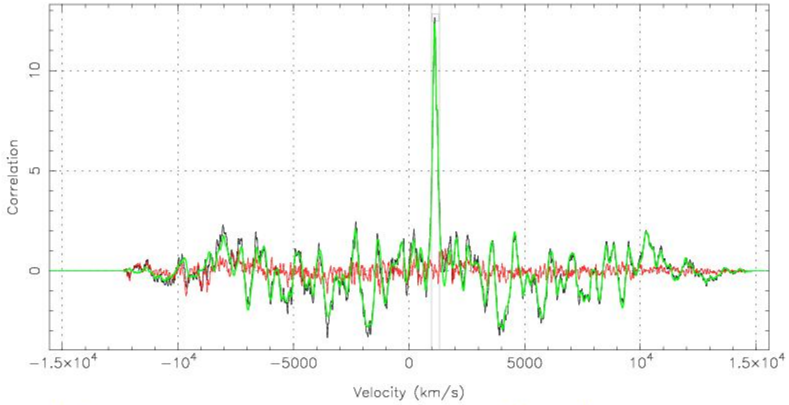


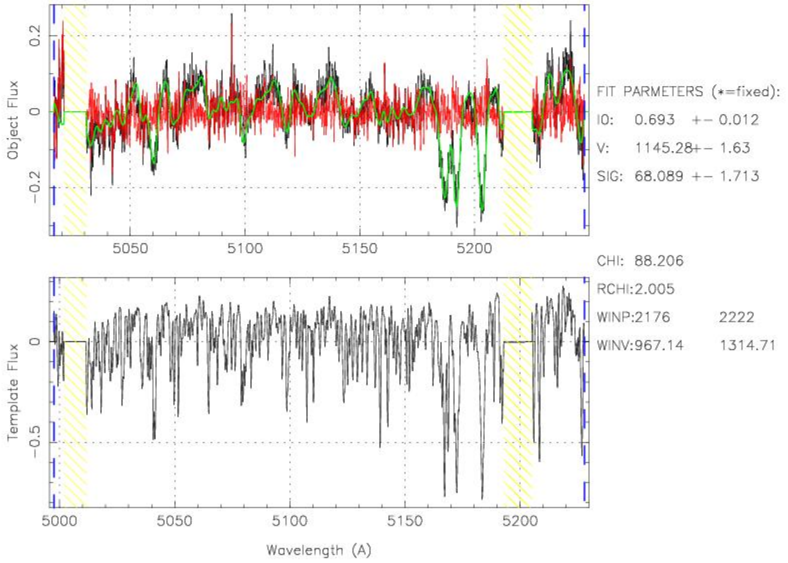
$V_{los} = V \sin i$

$\sigma_{los}^2 = \sigma_\phi^2 \sin^2 i + \sigma_z^2 \cos^2 i$

$V_{los} = 0$

$\sigma_{los}^2 = \sigma_R^2 \sin^2 i + \sigma_z^2 \cos^2 i$





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