A500 / Problem Set #5 / Solutions

1.a The central wavelength is found by solving for $\beta = 0$ from the given values for α, $\theta_{\rm cc}$ and their relation, and plugging this in to the grating equation for the given groove density and order. The groove spacing for $\sigma = 1000$ l/mm is 10^{-3} mm $= 1$ μm $= 1000$ nm. This yields $\lambda = 500$ nm or 0.5 μm or 5000 Å.

1.b The inverse linear dispersion at the camera focal surface ($d\lambda/dl$) is given by the inverse angular disperion ($1/\gamma = d\lambda/d\beta$) divided by the camera focal-length ($f_{\rm cam} = 250$ mm), i.e., $dl \ = \ f_{\rm cam}\ d\beta$. The angular dispersion is found from taking the derivative of the grating equation w.r.t. β. This yields $1/\gamma = \sigma \cos\beta$ for m=1. The linear dispersion is $(1000 \times \cos(0)/250)$ (nm/mm) = 4 nm/mm, or just 4×10^{-6}.

1.c The geometric demagnification is the ratio of camera to collimator focal-lengths, $250/1000 = 0.25$. The anamorphic factor is $r = \cos\alpha/\cos\beta = \cos(30)$. The reimaged slit-width in the spectral dimension is the monochromatic slit width. Since $\cos(30)$ in the anoamorphic factor cancels with same term in $231/\cos(30)$, the slit-width is just $0.25 \times 231/r = 0.25 \times 200\mu{\rm m} = 50\mu{\rm m}$.

1.d The spectral resolution at the central wavelength and slit center is simply the ratio of the central wavelength to the product of the linear dispersion and the monochromatic slit width: $R = 0.5\mu{\rm m}/(50\mu{\rm m} \times 4 \times 10^{-6}) \ = \ 0.25 \times 10^4 \ = \ 2500$.

1.e A source at $y' = 20$ mag has an equivalent photon flux of

$$f \ = \ f_0 \ \times \ {\rm dexp}(-0.4 \times y') \ \times \ \left(\frac{15.1}{10^{-6}}\right) \ \times \ \left(\frac{\Delta\lambda}{\lambda}\right)$$

$$= \ 2500 \ \times \ {\rm dexp}(-0.4 \times 20) \ \times \ 15.1 \times 10^6 \ / \ 2500 \ = \ 15.1 \times 10^{-2} \ {\rm photons\ m^{-2}\ s^{-1}}$$

where we have taken $\frac{\Delta\lambda}{\lambda} = 1/R$. Multiplying by 10 m^2 and 20% efficiency we find the photon count per spectral resolution element is about 0.3 e$^-$/s.

2.a On axis there is only diffraction, but we will assume this is negligible since the collimated beam is many orders of magnitude larger than the wavelength of light.

2.b Off-axis there is diffraction, coma, and astigmatism. Again, ignore diffraction. The ends of the slit are off-axis by $(100/2)/500 = 0.1$ rad. For the f/2 collimator we have $\delta_{\rm coma} = 0.3/64 \sim 0.5 \times 10^{-2}$ rad, and $\delta_{\rm astig} = 0.01/4 = 0.25 \times 10^{-2}$ rad. Summing in quadrature gives $\delta_{\rm total} \ = \ \sqrt{\sum_i \delta_i^2} \ = \ 0.5 \times 10^{-2} \ \times \sqrt{1 + 0.5^2} \ \sim \ 0.6 \times 10^{-2}$ rad.

2.c The physical size of the aberration that is relevant to comparing to the entrance slit (here the fiber diameter) is given by $\delta_{\rm total}$ times the collimator focal-length, 1000 mm. Since there are negligible aberrations at the center of the slit the equivalent fiber diameter is $\sim$0.

2.d At the end of the slit, $\delta_{\rm total} \ \sim \ 0.6 \times 10^{-2}$ rad times the collimator focal-length (500mm) gives the effective slit width of ~ 3 mm.

3.a WIYN Bench Spectrograph fibers have diameters ranging from 100 to 600 μm, all of which are much smaller than the $\sim$3 mm effective slit-width at the slit ends due to aberrations, e.g., thinking in terms of quadrature summation, $(0.6/3)^2 = 4 \times 10^{-2}$ is insignificant (just a 4% increase).

3.b No. While the camera focal length determines the physical spot size at the detector, it simply scales the effective angles introduced by the collimator aberrations and entrance slit.

3.c Anamorphism cannot change your conclusion because anamorphic factors simply scale the existing angular factors that contribute to $d\alpha$ to some other $d\beta$, but the relative contributions will remain the same.

4. A surface-brightness of 25 mag arcsec^{-2} (AB) in the V band corresponds to SB $= 3631 \times$ dexp$(-0.4 \times 25) \ = \ 0.363$ μJy arcsec^{-2}.

To convert to photons per pixel per second we use the flux conversion formula with the telescope area ($\mathrm{A_T} = 25\pi$), the system efficiency ($\epsilon = 1/\pi$), and the band-pass width of 0.16 nepers, but we also need to know the solid angle of a pixel. Call the latter Ω_p.

The focal plane scale is $f-$ratio$\times \mathrm{D_T}$ in units of distance per radian. Converting to mm per arcsec yields 3 $\times$10m $\times$1000(mm/m)/206265 = 0.145 mm / arcsec. That means there are 10×10 pixels per arcsec^{-2} so $\Omega_p = 0.01$ arcsec^2

Photons per pixel per second is then SB(μJy) $\times$ 15.1 $\times$ $\mathrm{A_T}$ $\times$ ϵ $\times$ nepers(V band) $\times$ Ω_p $=$ 0.363$\times$ 15.1 $\times$ 25π $\times$ 1/π $\times$ 0.16 $\times$ 0.01 $=$ 0.22.

5.a For this first part we don't care about how pixel sizes have changed with the spectrograph magnification, but we do need to recognize that (i) our solid angle is no longer Ω_p but simply $\Omega = 1$ arcsec^2, and (ii) the nepers in our band-pass is given by the spectral resolution elements, i.e., 1/R.

Repeating the calculation from (2) we have photons per pixel per second per resolution element is then SB(μJy)$\times$15.1$\times \mathrm{A_T} \times \epsilon$ $\times$ $(1/R)$(nepers) $\times$ Ω $=$ 0.363$\times$ 15.1 $\times$ 25π $\times$ 1/π $\times$ 1/5500 $\times$ 1 $\sim$ 0.025.

5.b Now all we have to do is to compute the number of pixels sampling the resolution element and 1 arcsec along the slit length. Luckily we have a slit-width of 1 arcsec and a Littrow configuration. The pixels are simply demagnified in each of two dimension by the camera-collimator focal-length ratio which is the same as the ratio of their f-ratios, i.e., 1.5/3 = 0.5. Hence the number of pixels per resolution element and along 1 arcsec of the slit length is reduced from 100 to 25, which is the same as $\Omega_p = 0.04$. We then compute the photons per pixel per second per pixel to be 0.025/25 $\sim$ 0.001.

6. Using the resolution formula in Littrow configuration yields $R \equiv \lambda/\delta\lambda = 2(f_{\mathrm{coll}}/w_{\mathrm{slit}}) \tan\alpha$. The slit-width (1 arcsec) is found from Problem-4 to be 0.145 mm. This means $\tan\alpha = (5500/2) \times (0.145/1e3) \ = \ 0.399$, so that $\alpha = 21.7$ deg.

Using the grating equation at this angle for $\alpha = \beta = 21.7$ at $\lambda = 550$ nm and m = 1 yields a groove spacing $\sigma \ = \ \lambda/(2 \times \sin\alpha) = 742.45$ nm, i.e., 1347 l/mm.