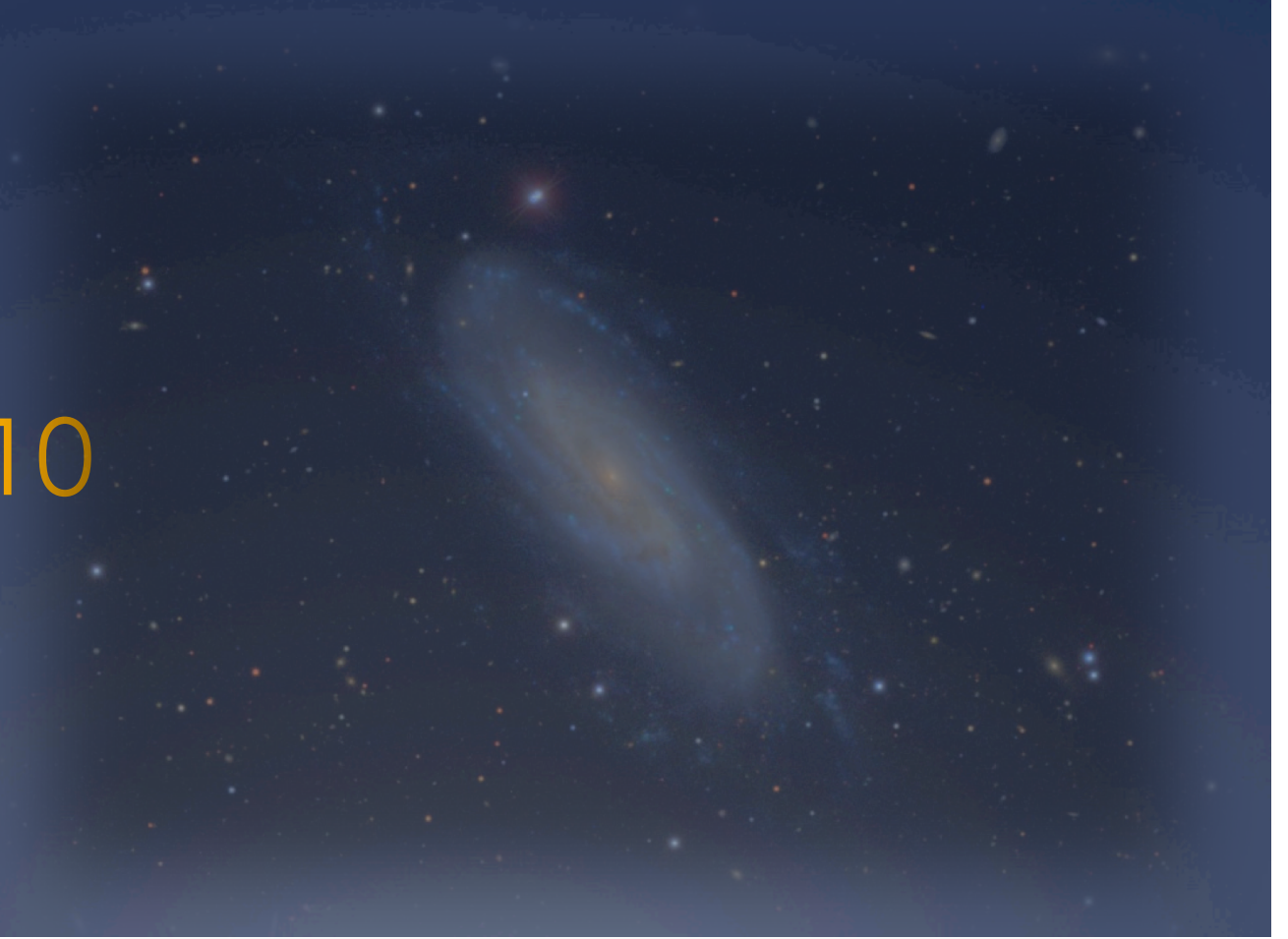


Astronomy 330

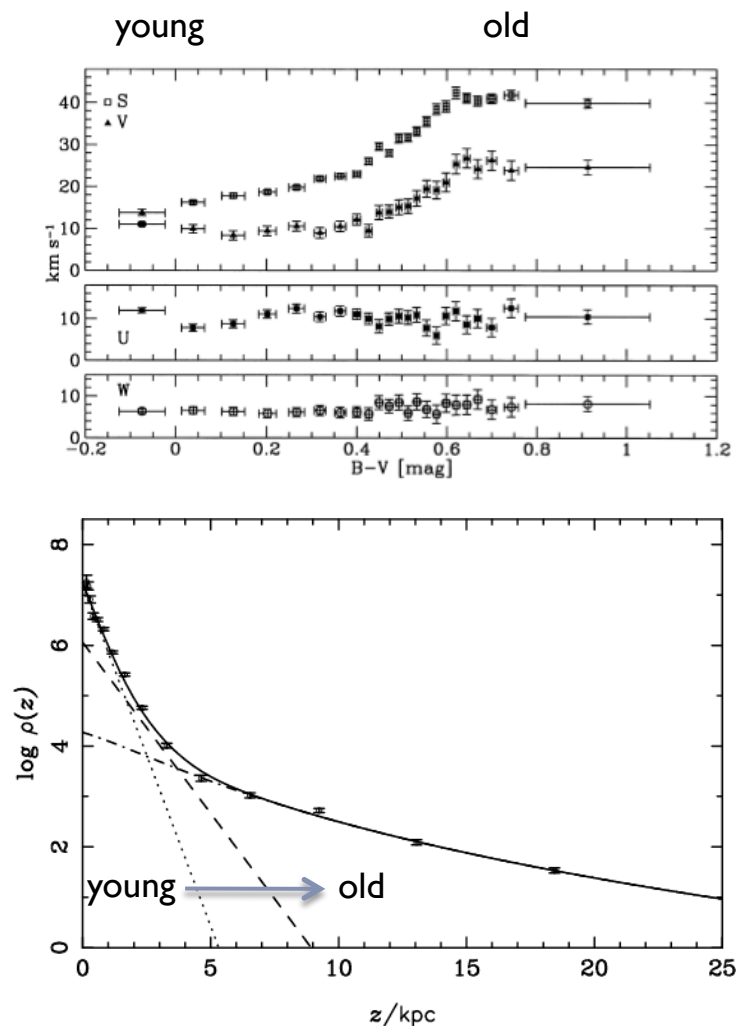
Lecture 9

01 Oct 2010



Outline

- ▶ MW kinematics
 - ▶ Disk
 - ▶ Halo
- ▶ Galactic dynamics
 - ▶ Potentials
 - ▶ Energetics
 - ▶ Flat rotation curves



Galactic Dynamics

- ▶ Basic morphology of galaxies (and parts of galaxies) is determined by the orbits of stars
 - ▶ disk galaxies are disk-like because most of the stars orbit in nearly circular orbits in a flattened plane.
- ▶ What determines the stellar orbits? The gravitational potential: $\Phi(r, \theta, z)$.
- ▶ What determines the gravitational potential? The distribution of mass, $\rho(r, \theta, z)$.



Fundamentals: Gravitational Potentials

- ▶ Newton's gravitational force law for a point-mass M
 - ▶ $d(m\mathbf{v})/dt = -GmMr/r^3$
 $= -m \nabla \Phi(r)$
 - ▶ \mathbf{v}, \mathbf{r} vectors; r scalar; ∇ the gradient
 - ▶ Φ is the gravitational potential, $\Phi = -GM/r$
 - ▶ Thus, $\mathbf{F}(\mathbf{x}) = -\nabla \Phi$
 - ▶ the force is determined by the gradient of the potential.
- ▶ Gravitational potential generalized:
 - ▶ $\Phi(\mathbf{x}) \equiv -G \int (\rho(\mathbf{x}')/|\mathbf{x}'-\mathbf{x}|) d^3\mathbf{x}'$
 - ▶ $\mathbf{F}(\mathbf{x}) = G \int [(\mathbf{x}'-\mathbf{x})/|\mathbf{x}'-\mathbf{x}|^3] \rho(\mathbf{x}') d^3\mathbf{x}'$
 - ▶ Force on a unit mass at position, \mathbf{x} , from a distribution of mass $\rho(\mathbf{x})$.
- ▶ Take the divergence of $\mathbf{F}(\mathbf{x})$ [$\nabla \cdot \mathbf{F}(\mathbf{x}) = -\nabla^2 \Phi(\mathbf{x})$] to get Poisson's equation:

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x})$$

- ▶ Directly related to Gauss's law:
 - ▶ In the absence of sources: $\nabla \cdot \mathbf{F}(\mathbf{x}) = 0$
 - ▶ Laplace equation: $\nabla^2 \Phi(\mathbf{x}) = 0$

See S&G and divergence theorem for derivation. Think:
What does “divergence” mean?



Fundamentals: Divergence theorem

- ▶ Divergence theorem states that for some vector \mathbf{F}

- ▶ $\int \nabla \cdot \mathbf{F} \, dV = \int \mathbf{F} \cdot d\mathbf{S}$

- ▶ Consider volume to be subdivided into a large number of small cells with volume ΔV_i .
- ▶ For the cell-walls bounded by the surface, the sum of the surface-integrals for these cell-walls equals the surface-integral for the volume.
 - $\sum_i \int \mathbf{F} \cdot d\mathbf{S}_i = \int \mathbf{F} \cdot d\mathbf{S}$
- ▶ For the remainder of surfaces, since the outward surface-normal of one cell is opposite that of the surface of the adjacent cell, the surface integrals cancel.
- ▶ We can also write:
 - $\sum_i \left[(1/\Delta V_i) \int \mathbf{F} \cdot d\mathbf{S}_i \right] \Delta V_i = \int \mathbf{F} \cdot d\mathbf{S}$
- ▶ In the limit where $\Delta V_i \rightarrow 0$, the sum of the surface-integrals becomes an integral over V
- ▶ the ratio of the surface-integrals to ΔV_i as $\Delta V_i \rightarrow 0$ is the divergence of \mathbf{F} .



Divergence theorem corollary

- ▶ For scalar and vector functions g and \mathbf{F} :
 - ▶ $\int g \nabla \cdot \mathbf{F} \, dV = \int g \mathbf{F} \cdot d\mathbf{S} - \int (\mathbf{F} \cdot \nabla) g \, dV$



Application of potentials to galaxies

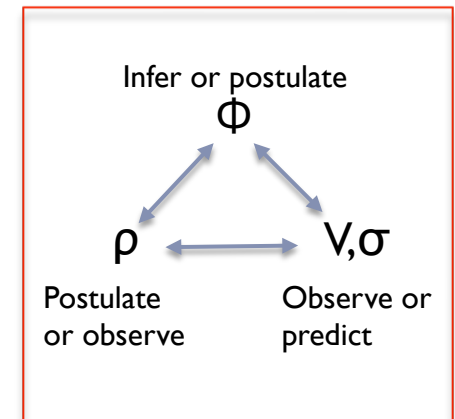
- ▶ Here's the process:

- ▶ We start by looking at some very simple geometric cases
- ▶ Define a few terms that help us think about and characterize the potentials
- ▶ Become more sophisticated in the form of the potential to be more realistic in matching galaxies

- ▶ Concepts:

- ▶ circular and escape velocities
- ▶ Time scales: dynamical, free-fall
- ▶ Potential (Φ or PE) and kinematic energy (K or KE)
- ▶ Energy Conservation and Virial Theorem
- ▶ Angular momentum

- ▶ Example: rotation curves of galaxies



Energy considerations

- ▶ Recall:

- ▶ $d(m\mathbf{v})/dt = -m \nabla \Phi(\mathbf{x})$,

- ▶ Take the scalar product with \mathbf{v}

- ▶ $\mathbf{v} \cdot d(m\mathbf{v})/dt + m\mathbf{v} \cdot \nabla \Phi(\mathbf{x}) = 0$

- ▶ $\rightarrow d/dt [\frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})] = 0$

- where $d\Phi(\mathbf{x}) / dt = \mathbf{v} \cdot \nabla \Phi(\mathbf{x})$

- ▶ Total energy defined:

- ▶ $E = KE + PE = \frac{1}{2} m\mathbf{v}^2 + m\Phi(\mathbf{x})$

- ▶ *This means E is constant for closed system*

- ▶ e.g., an unperturbed orbit of a star

- ▶ This is true for static potentials.

- ▶ If there is a time varying potential (i.e. in a cluster) only the total energy is conserved (not the energy of an individual star)

- ▶ For an external force add the summation of $\mathbf{F}_{\text{ext}} \cdot \mathbf{x}$



Kinetic energy and escape velocity

- ▶ If E is constant for closed system
- ▶ and by definition: $KE \geq 0$
 - ▶ Implications:
 - ▶ As $x \rightarrow \infty$ (far from potential) $\Phi(x) \rightarrow 0$.
 - ▶ If $E > 0$ at $x=\infty$ then $v > 0$
 - ▶ i.e., the object has escaped the potential
 - ▶ Escape velocity for critical energy ($E=0$):
 - ▶ $v_e(\mathbf{x}) = (2|\Phi(\mathbf{x})|)^{1/2}$



Potential Energy

- ▶ Work (W) done in assembling a mass distribution is the potential energy.
- ▶ Start with initial portion of mass $\rho(\mathbf{x})$ which generates potential $\Phi(\mathbf{x})$
- ▶ Add an increment of mass δm : the work done is $\delta m \Phi(\mathbf{x})$
 - ▶ The work per unit mass over a distance \mathbf{x} is
 - ▶ $\mathbf{F} \cdot \mathbf{x} = -\mathbf{x} \cdot \nabla \Phi(\mathbf{x})$
 - ▶ Integrating the work from $\mathbf{x}=\infty$ to some finite distance, where $\Phi(\mathbf{x} \rightarrow \infty) \rightarrow 0$
implies $\Phi(\mathbf{x})$ as the total work (potential energy) per unit mass.
- ▶ Think of δm as equivalent to a change in density over the assembled volume:
 - ▶ $\int \delta \rho(\mathbf{x}) d^3\mathbf{x}$.
- ▶ Then work done, $\delta m \Phi(\mathbf{x})$ is:
 - ▶ $\delta W = \int \delta \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}$.



Potential Energy (continued)

- ▶ Apply Poisson's equation on $\delta \rho$ yields
 - ▶ $\delta W = (1/4\pi G) \int \Phi(\mathbf{x}) \nabla^2(\delta \Phi) d^3\mathbf{x}$.
- ▶ Use the divergence theorem to write:
 - ▶ $\delta W = (1/4\pi G) \int \Phi(\mathbf{x}) \nabla(\delta \Phi) \cdot d\mathbf{S} - (1/4\pi G) \int \nabla \Phi(\mathbf{x}) \cdot \nabla(\delta \Phi) d^3\mathbf{x}$
- ▶ The surface-integral vanishes because:
 - ▶ $\Phi(\mathbf{r})$ and $|\nabla(\delta \Phi)|^{1/2}$ go as r^{-1} as $r \rightarrow \infty$
 - ▶ i.e, the integrand goes as r^{-3} while the surface area goes as r^2
- ▶ Also there is this identity: $\nabla \Phi(\mathbf{x}) \cdot \nabla(\delta \Phi) = \frac{1}{2} \delta |(\nabla \Phi)|^2$
- ▶ From which it follows:
 - ▶ $\delta W = -(1/8\pi G) \delta \left[\int |\nabla \Phi|^2 d^3\mathbf{x} \right]$
- ▶ Sum over all δW to get
 - ▶ $W = -(1/8\pi G) \int |\nabla \Phi|^2 d^3\mathbf{x}$
- ▶ Again apply Divergence theorem and Poisson equation to arrive at

$$W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}$$



Virial Theorem

- ▶ In an isolated system composed of multiple mass units, these masses can change (and exchange) their kinetic and potential energy as long as
 - ▶ the sum is constant (E is conserved overall):
 - ▶ $\langle KE \rangle + \langle PE \rangle = \text{constant}$
 - where $\langle \rangle$ are averages over the system
 - ▶ If the system is in equilibrium, the kinetic energy must balance the potential energy, e.g., an object in a circular orbit around a fixed mass.
 - ▶ This equilibrium configuration is referred to as “virialization”, and implies:
 - ▶ $2\langle KE \rangle + \langle PE \rangle = 0$
 - ▶ For an external force add the summation of $\mathbf{F}_{\text{ext}} \cdot \mathbf{x}$:
 - ▶ $2\langle KE \rangle + \langle PE \rangle + \mathbf{F}_{\text{ext}} \cdot \mathbf{x} = 0$



Virial Theorem in practice

- ▶ This is a primary tool for inferring masses of dynamical systems in astronomy, so it is important. Here's how it is applied:
 - ▶ Measure the velocity dispersion of some spherical system
 - ▶ (e.g., a star cluster).
 - ▶ We observe V_r , from which we can derive: $\sigma_r = \langle V_r \rangle^{1/2}$
 - ▶ If we assume motions are isotropic, then $\mathbf{V} \cdot \mathbf{V} = \sum_i V_i^2 \sim 3 \sigma_r^2$.
 - ▶ Kinetic energy = $(3/2) M \sigma_r^2$
 - ▶ Potential energy looks something like: $-GM^2/2fr_c$
 - ▶ r_c is a “core radius” at which point the surface brightness is $1/2$ of its central value.
 - ▶ f is a fudge-factor or order unity that accounts for the details of the mass distribution in the integral $\int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}$.
 - ▶ Solve for mass, M



Angular Momentum, Torque & Integrals of Motion

- ▶ $\mathbf{L} \equiv \mathbf{x} \times m\mathbf{v}$
- ▶ $\mathbf{N} \equiv \text{torque} = \mathbf{x} \times \mathbf{F} = d\mathbf{L}/dt = -m\mathbf{x} \times \nabla \Phi$
 - ▶ Torque is rate of change of angular momentum
- ▶ For a spherically symmetric galaxy, L is constant. For an axisymmetric galaxy (most galaxies), only the component parallel to the symmetry axis is constant
- ▶ Integrals of motion:
 - ▶ Any function of phase-space coordinates (\mathbf{x}, \mathbf{v}) that is constant along an orbit
 - ▶ In static potential $\Phi(\mathbf{x})$:
 - ▶ $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})$ is an integral of motion
 - ▶ If $\Phi(R, z, t)$ is axisymmetric about z-axis:
 - ▶ L_z is an integral of motion
 - ▶ In spherical potential $\Phi(R, t)$:
 - ▶ all three components of L are integrals of motion.



Spherical mass distributions

- ▶ *Start simple....*
- ▶ Newton showed:
- ▶ A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.
 - ▶ Mass contained in solid-angle $\delta\Omega$ of shell as seen by body depends on distance to shell:
 - ▶ $\delta m = \Sigma \delta\Omega \times r^2$, where Σ is the mass-surface-density of the shell.
 - ▶ Hence in any two directions:
 - $\delta m_1 / \delta m_2 = (r_1 / r_2)^2 \rightarrow \delta F_1 = - \delta F_2$
 - particle is attracted equally in opposite directions
 - ▶ $\nabla \Phi = -F = 0$
- ▶ The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.
 - ▶ $\Phi = -GM/R$



Spherical distributions: characteristic velocities

- ▶ The gravitational attraction of a density distribution, $\rho(r')$, on a particle at distance, r , is:
 - ▶ $F(r) = -(d\Phi/dr) = -GM(r)/r^2$
 - ▶ $M(r) = 4\pi \int \rho(r') r'^2 dr'$
- ▶ Circular speed:
 - ▶ In any potential $d\Phi/dr$ is the radial acceleration
 - ▶ For a circular orbit, the acceleration is v^2/r
 - ▶ \rightarrow $v_c^2 = r(d\Phi/dr) = GM(r)/r$
 - ▶ outside a spherical mass distribution, v_c goes as $r^{1/2}$
 - Keplerian
- ▶ Escape speed: $v_e(r) = (2|\Phi(r)|)^{1/2} = [2 \int GM(r) dr/r^2]^{1/2}$



Homogeneous Sphere: characteristic time-scales

- ▶ $M(r) = (4/3)\pi r^3 \rho$
 - ▶ ρ is constant
- ▶ For particle on circular orbit, $v_c = (4\pi G \rho / 3)^{1/2} r$
 - ▶ rises linearly with r .
 - ▶ Check out the Galaxy's inner rotation curve.
 - ▶ What does this say about the bulge?
- ▶ Orbital period: $T = 2\pi r / v_c = (3\pi / G \rho)^{1/2}$
- ▶ Now release a point mass from rest at r :
 - ▶ $d^2r/dt^2 = -GM(r)/r^2 = -(4\pi G \rho / 3)r$
 - ▶ Looks like the eqn of motion of a harmonic oscillator with frequency $= 2\pi/T$
 - ▶ Particle will reach $r = 0$ in 1/4 period ($T/4$), or
- ▶ $t_{\text{dyn}} \equiv (3\pi / 16G \rho)^{1/2}$



Isochrone Potential

- ▶ *Since nothing is really homogeneous...*
- ▶ $\Phi(r) = -GM/[b+(b^2+r^2)^{1/2}]$
 - ▶ b is some constant to set the scale
 - ▶ $v_c^2(r) = GMr^2/[(b+a)^2a] \rightarrow (GM/r)^{1/2}$ at large r
 - ▶ $a \equiv (b^2+r^2)^{1/2}$
- ▶ This simple potential has the advantage of having constant density at small r , falling to zero at large r
 - ▶ $\rho_0 = 3M / 16\pi Gb^3$
- ▶ Similar to the so-called Plummer model used by Plummer (1911) to fit the density distribution of globular clusters:
 - ▶ $\Phi(r) = -GM / (b^2+r^2)^{1/2}$
 - ▶ $\rho(r) = (3M / 4\pi Gb^3) (1+r^2/b^2)^{-5/2}$



Singular Isothermal Sphere

- ▶ Physical motivation:
 - ▶ Hydrostatic equilibrium: pressure support balances gravitational potential
 - ▶ $dp/dr = (k_B T/m) d\rho/dr = -\rho GM(r)/r^2$
 - ▶ $\rho(r) = \sigma^2/2\pi G r^2$
 - where $\sigma^2 = k_B T/m$
- ▶ Singular at origin so define characteristic values:
 - ▶ $\rho' = \rho/\rho_0$
 - ▶ $r' = r/r_0$
 - ▶ $r_0 \equiv (9\sigma^2/4\pi G\rho_0)^{1/2}$
- ▶ $\Phi(r)$ is straight-forward to derive given our definitions:
 - ▶ $\Phi(r) = V_c^2 \ln(r/r_0)$
 - ▶ $V_c^2 = 4\pi\rho_0 r_0^2$
- ▶ A special class of power-law potentials for $\alpha=2$
 - ▶ $\rho(r) = \rho_0 (r_0/r)^\alpha$
 - ▶ $M(r) = 4\pi\rho_0 r_0^\alpha r^{(3-\alpha)}/(3-\alpha)$
 - ▶ $V_c^2(r) = 4\pi\rho_0 r_0^\alpha r^{(2-\alpha)}/(3-\alpha)$

Look what happens to $V(r)$ when $\alpha=2$

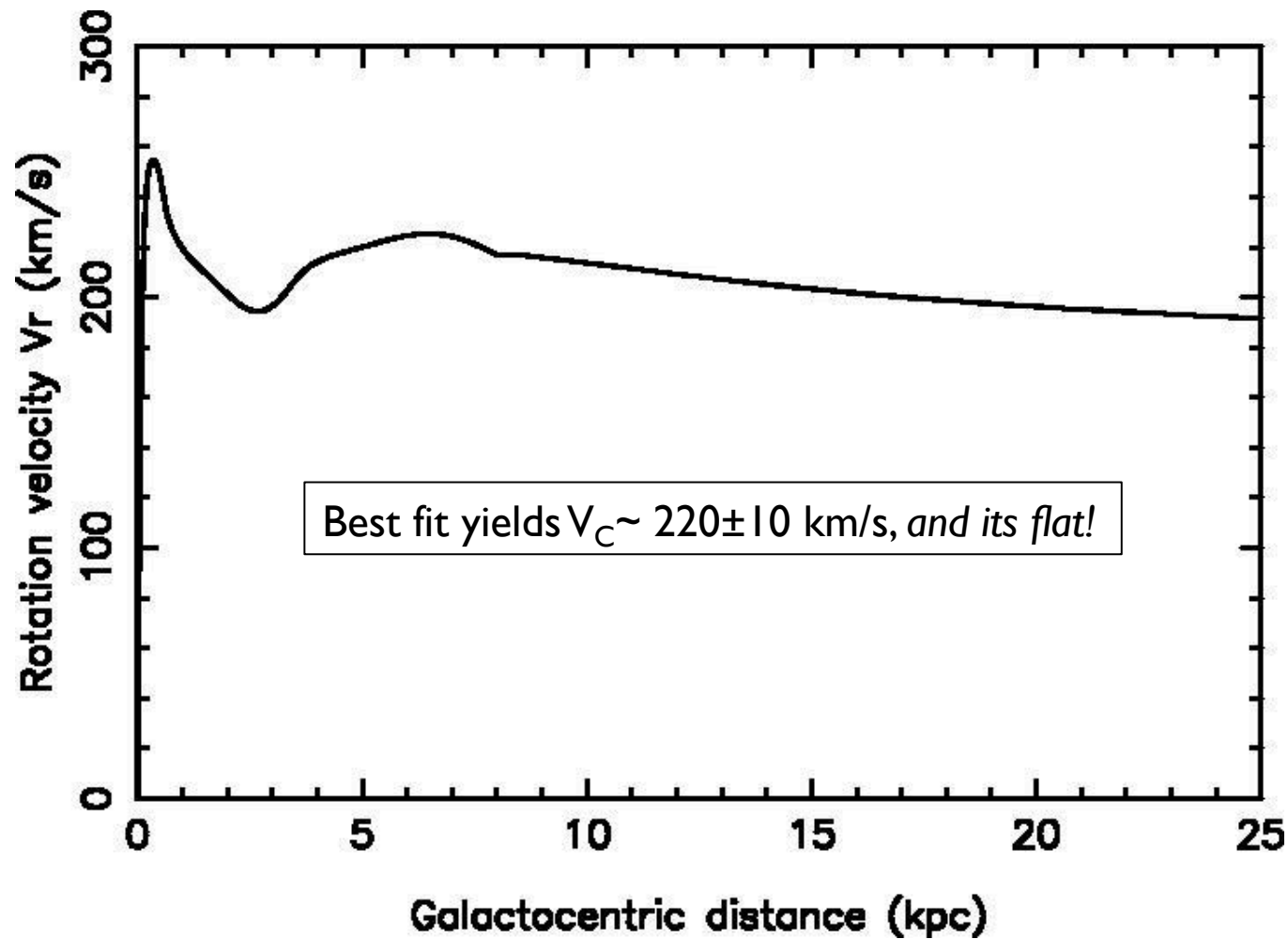


Pseudo-Isothermal Sphere

- ▶ Physical motivation: avoid singularity at $r=0$, but stay close to functional form. Posit:
 - ▶ $\rho(r) = \rho_0[1 + (r/r_c)^2]^{-1}$
- ▶ $\Phi(r)$ is straight-forward to derive given our definitions
- ▶ $V(r) = (4\pi G \rho_0 r_c^2 [1 - (r_c/r)\arctan(r/r_c)])^{1/2}$
 - ▶ This gives a good match to most rotation curves within the optical portion of the disk.
 - ▶ But it does not give a good description of the light distribution of disks.



Flat rotation curves: the Milky Way



Flat rotation curves: external galaxies

Which looks most like the MW?
Why different shapes and extents?

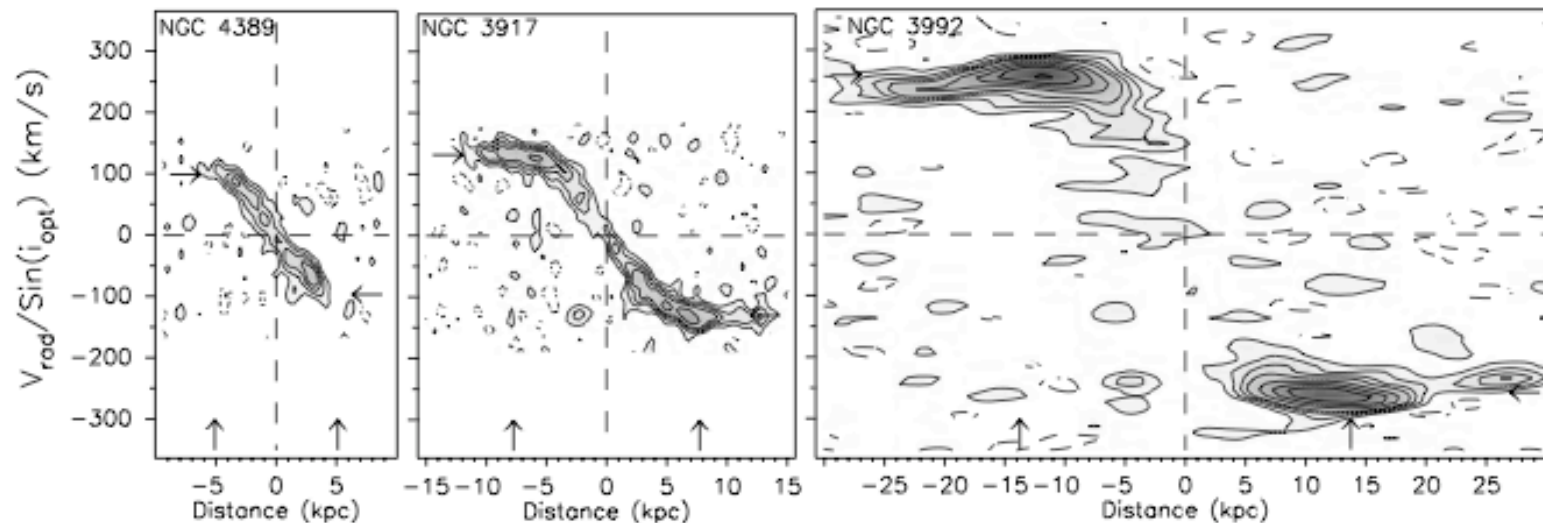
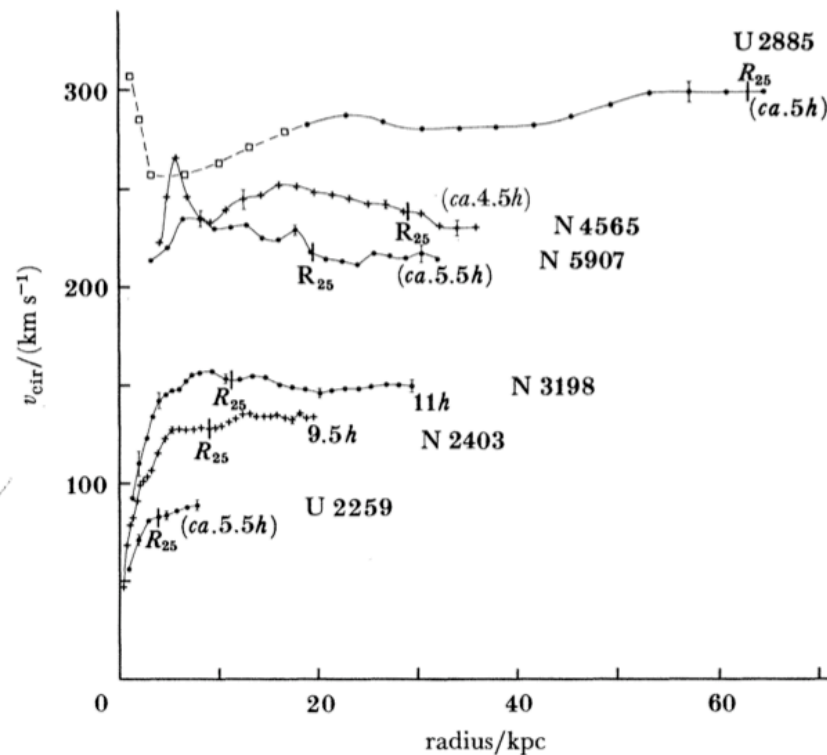


FIG. 2.—Prototype examples of the three categories of rotation curves. *Left*: Galaxy with a rotation curve that rises continuously until the last measured point. The measured maximum rotational velocity V_{max} is set by the extent of the H I disk (*R* curve). *Middle*: “Classical” rotation curve; a gentle rise in the central regions with a smooth transition into the extended flat part (*F* curve). *Right*: Rotation curve that reaches a maximum in the optical regions after which it declines somewhat to an extended flat part in the outer disk. In this case, the maximum rotation velocity exceeds the amplitude of the flat part (*D* curve). The vertical arrows indicate $\pm R_{25}$, and the horizontal arrows indicate the rotational velocities as inferred from the global profiles.

Verheijen 2005, ApJ, 563, 694

Flat rotation curves: external galaxies



Van Albada et al.
1986, *Phil. Trans.*
Royal Soc. London,
320, 1556, 447

FIGURE 2. HI rotation curves for a number of spiral galaxies (Sancisi & van Albada 1986). Distances are based on $H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The optical radius, R_{25} , and the number of disc scalelengths, h , at the last measured point are indicated. For the inner region of UGC 2885 optical velocities (Rubin *et al.* 1986) have been used. All curves remain approximately flat beyond the turnover radius of the disc ($2.5 h$).

Flat rotation curves: the disk

- ▶ Disk component
- ▶ $\Sigma(r) = \Upsilon \times \mu(r)$
 - ▶ Σ is the mass surface-density
 - ▶ Υ is the mass-to-light ratio (M/L)
 - ▶ μ is the surface-brightness
 - ▶ Surface mass density ($M_{\odot} \text{ pc}^{-2}$) is just the mass to light ratio times the surface brightness ($L \text{ pc}^{-2}$)
- ▶ **Mass \rightarrow potential \rightarrow circular velocity**
 - ▶ The trick here is to deal with the non-circular density distribution.



Flat rotation curves: the exponential disk

- ▶ $\Sigma(r) = \Sigma_0 \exp(-r/h_R)$

- ▶ Mass:

- ▶ $M(r) = 2\pi \int \Sigma(r') r' dr' = 2\pi \Sigma_0 h_R^2 [1 - \exp(-r/h_R)(1 + r/h_R)]$

- ▶ **→ potential**

- ▶ $\Phi(r, z=0) = -\pi G \Sigma_0 r [I_0(y)K_0(y) - I_1(y)K_1(y)]$


- ▶ $y = r/2h_R$

- ▶ I, K are modified Bessel functions of the 1st and 2nd kinds.

- ▶ **→ circular velocity**

- ▶ $V_c^2(r) = r d\Phi/dr = 4\pi G \Sigma_0 h_R y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)]$

- ▶ Note: This is for an infinitely-thin exponential disk. In reality, disks have a thickness with axis ratios $h_R:h_z$ between 5:1 and 10:1

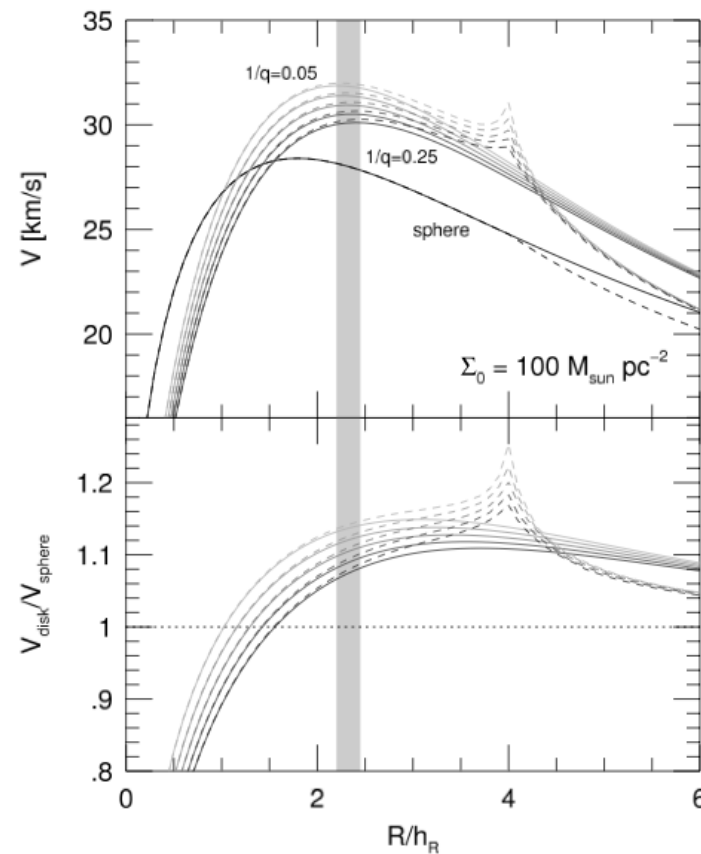


A bit of work; see
Freeman (1970) and
Toomre (1963)



Rotation from an exponential disk

Disk oblateness:
 $q = h_z/h_r$



← It isn't flat

↓ It is pretty flat

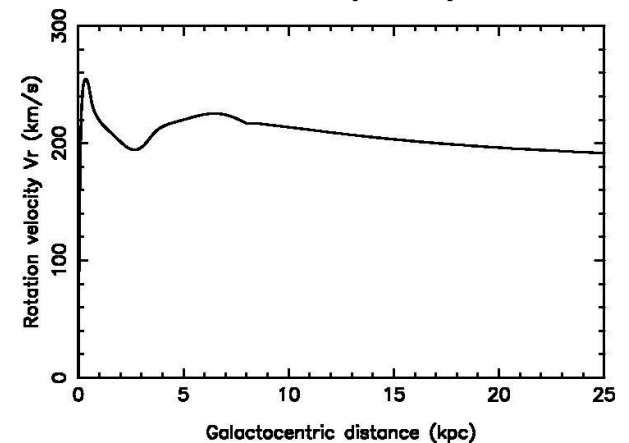


Fig. 17.— Rotation speed of an exponential disk with central mass surface density of $100 M_{\odot} \text{ pc}^{-2}$ and oblateness $0.05 < q < 0.25$ versus radius normalized by scale-length, compared to a spherical density distribution with the same enclosed mass. Bottom panel shows the ratio of spherical to disk velocities. Dashed and solid lines show disks truncated at $R/h_R=4$ and 10, respectively. The radial range where these disks have peak velocities is shaded in gray.

Flat rotation curves: the halo

- ▶ “Halo” component – remember we need $V(r)$ to be constant at large radius.
- ▶ One option is the singular isothermal sphere, here $V(r)$ is constant at all radii.
 - ▶ Is that plausible given observed rotation curves (e.g, MW)?
- ▶ Another option: the pseudo-isothermal sphere
 - ▶ $V(r) = (4\pi G \rho_0 r_c^2 [1 - (r_c/r)\arctan(r/r_c)])^{1/2}$
 - ▶ This gives a good match to most rotation curves within the optical portion of the disk.
- ▶ Also NFW motivated by CDM simulations (see S&G p. 117)

