Astronomy 330

Lecture 14

20 Oct 2010

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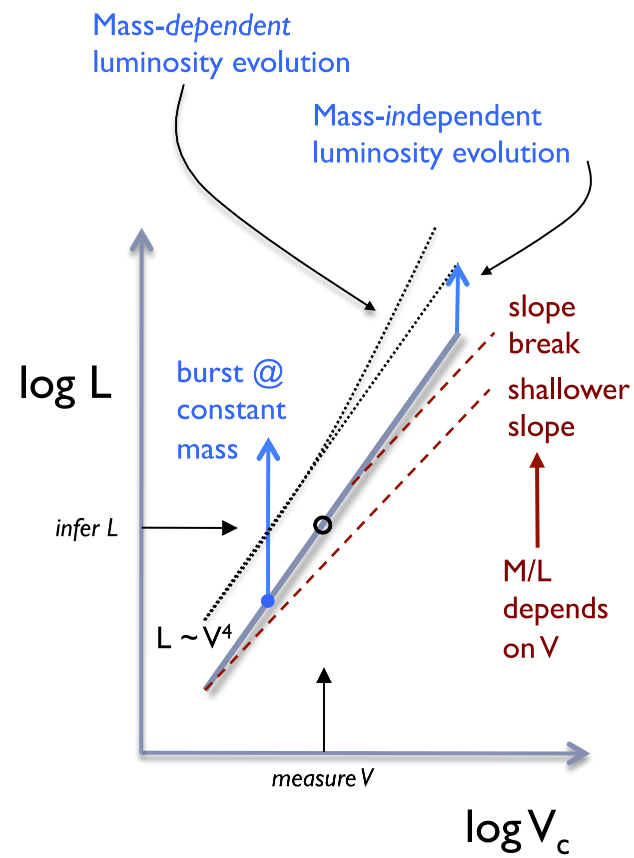
Outline

- Review:
  - Scaling relations
    - Tully-Fisher:
      - Why so good?
      - An evolutionary diagnostic and cosmological tool
- Continue with:
  - Dynamics of collisionless systems



Recall: Dynamics of collisionless systems

- Motivation:
  - Circular rotation is too simple and v_c gives us too little information to constrain Φ and hence ρ (e.g., rotation curves)
  - Without Φ and hence ρ we can't understand how mass has assembled and stars have formed
    - We can't even predict how the Tully-Fisher relation should evolve
  - Gas is messy because it requires understanding hydrodynamics, and likely magneto-hydrodynamics.
  - At are disposal are stars, nearly collisionless tracers of Φ!

Recall: Dynamics of collisionless systems

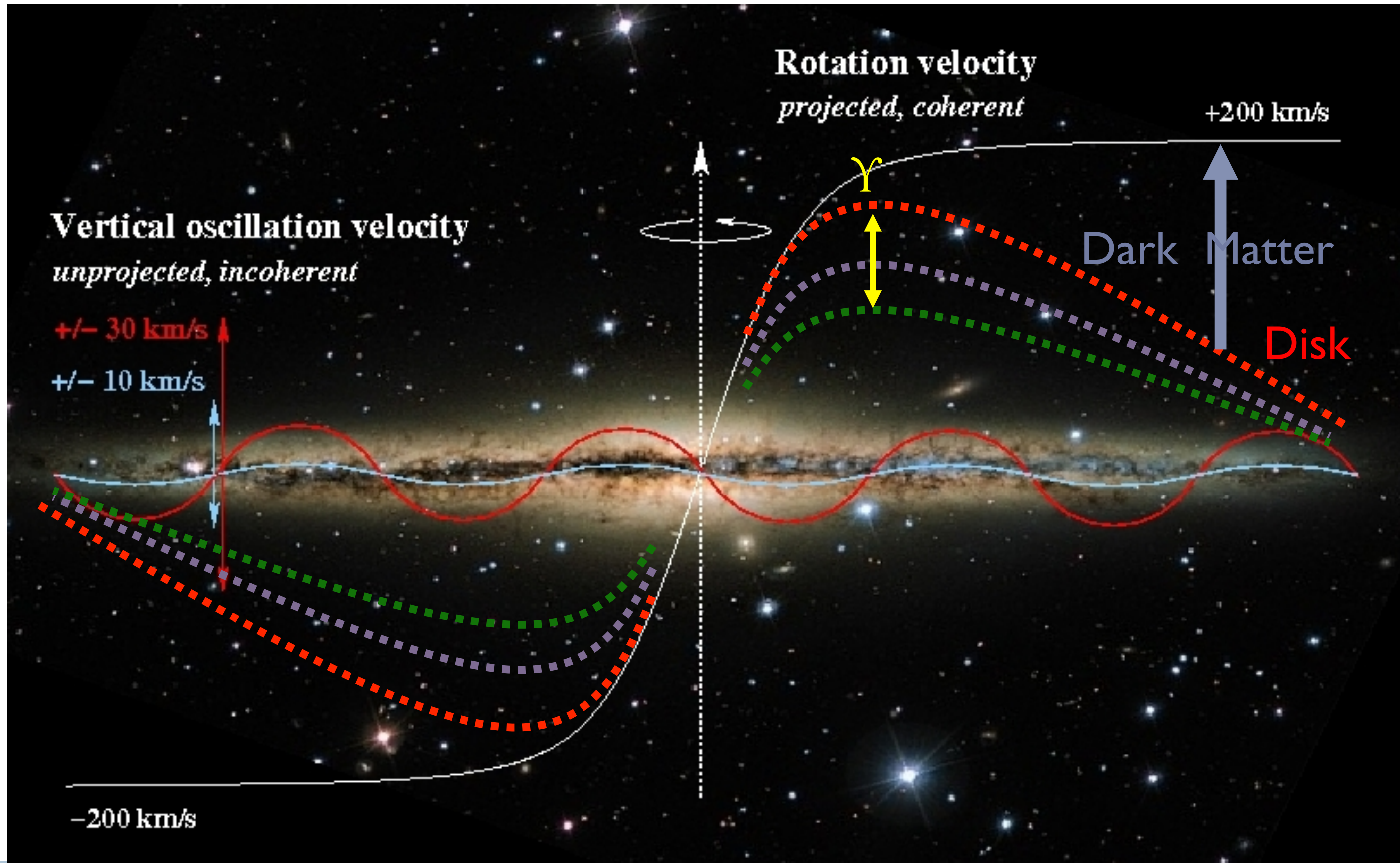
- How we'll proceed:
  - Start with the Continuity Equation (CE)
  - Use CE to motivate the Collissionless Bolztmann Equation (CBE), like CE but with a force term (remember $\nabla\Phi(\mathbf{x})$!)
  - Develop moments of CBE to relate v and σ and higher-order moments of velocity to Φ and ρ.
- Applications to realistic systems and real problems
  - Velocity ellipsoid
  - Asymmetric drift

Don't be intimidated by moment-integrals of differential equations in cylindrical coordinates: follow the terms, and look for physical intuition.

Disk heating
Disk mass
Disk stability

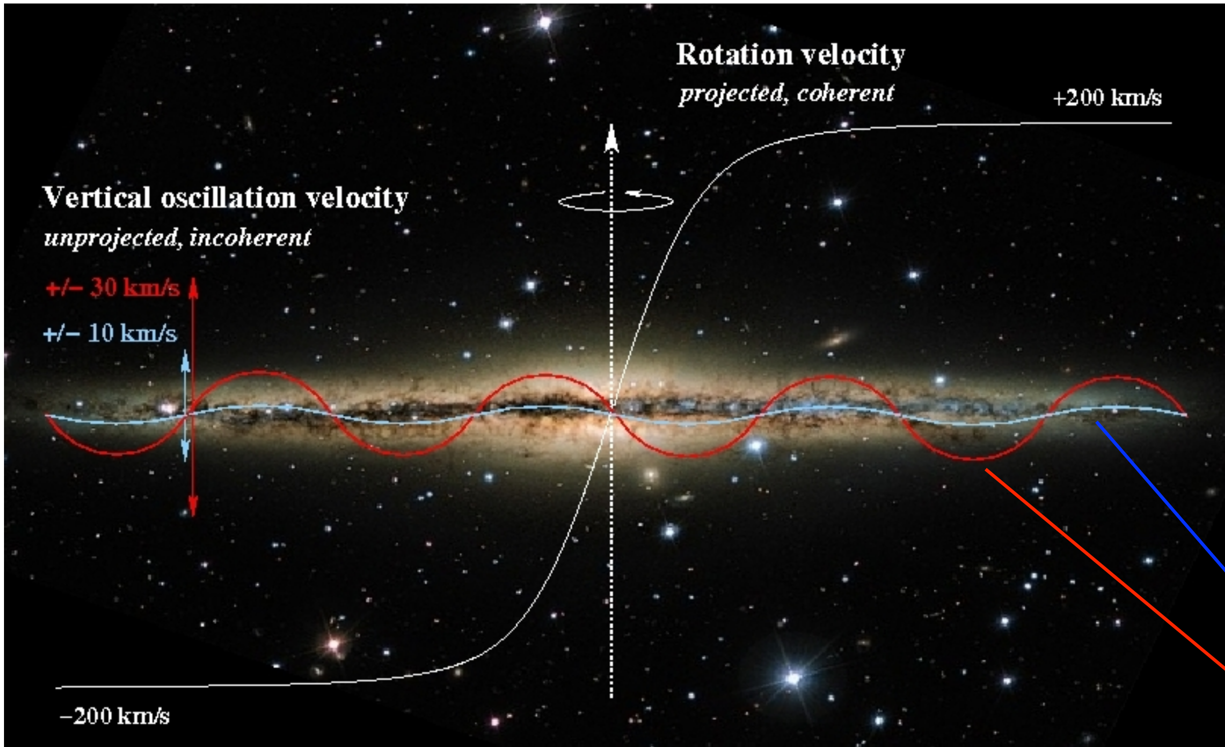
Example: Breaking the Disk-Halo Degeneracy

- Rotation provides the *total* mass within a given radius.
- Vertical oscillations of disk stars provides *disk* mass within given height

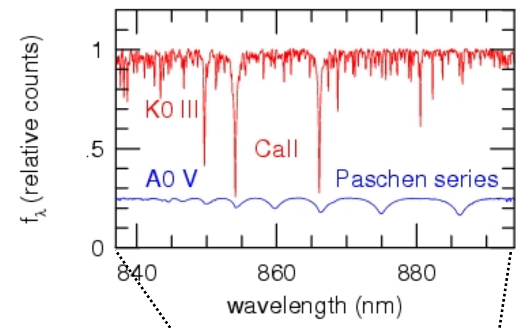


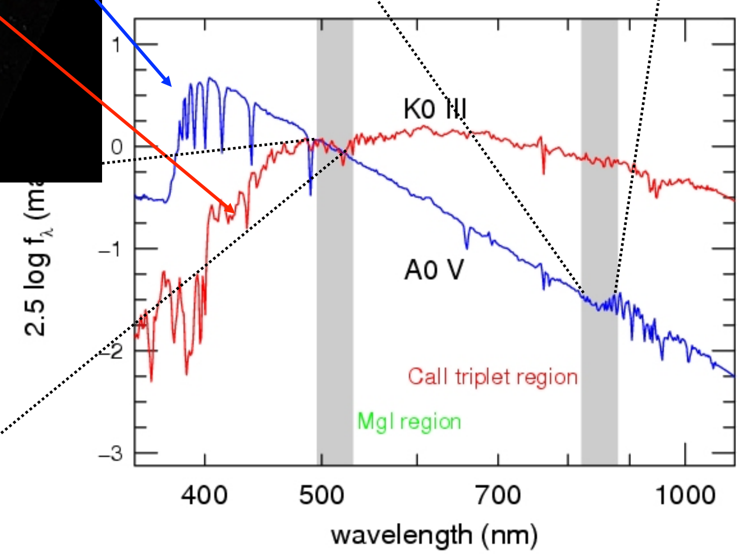
Vertical oscillations: a direct, dynamical approach

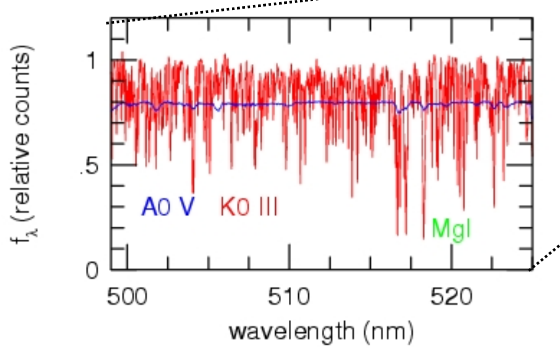
The kinematic signal



$\lambda/\delta\lambda$ = 11,000







- Young stars
  - Hot: weak or intrinsically broad lines
  - Dyamically cold, thin layer (extinction)
- Old stars
  - Cool: many strong, narrow lines
  - Dynamically warm, thick layer

Disk Mass formula

vertical distribution*

thickness

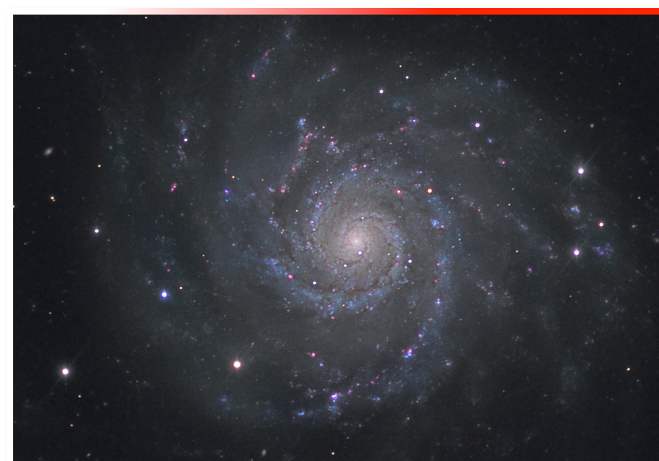
vertical oscillations

Use *statistical* measure of *disk thickness* from edge-on galaxies …

$$\Sigma = 100\left(\frac{k}{3/2}\right)^{-1}\left(\frac{h_z}{444\ \mathrm{pc}}\right)^{-1}\left(\frac{\sigma_z}{30\,\mathrm{km/s}}\right)^{2} \mathrm{M_{sol}\ pc^{-2}}$$

Disk mass surface density

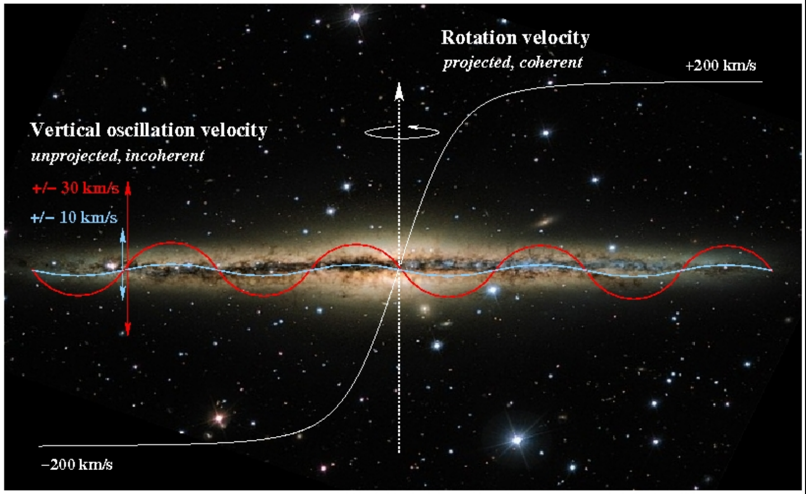
….and apply relation to face-on galaxies where the *vertical oscillations* of stars can be measured.



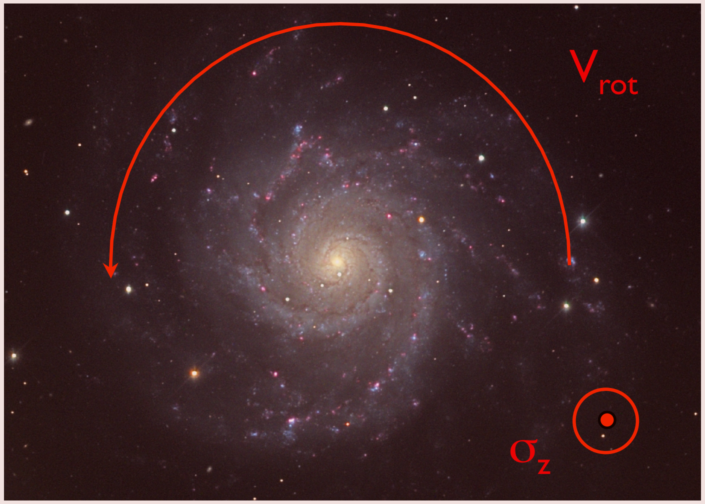
- * 1.5<k<2 for exp, sech, $sech^2$

Edge-on or Nearly Face-on ?

- ✓ Rotation projected
- × Vertical dispersion *inaccessible* except via statistical *kinematic* correlations
- ✓ Vertical height projected



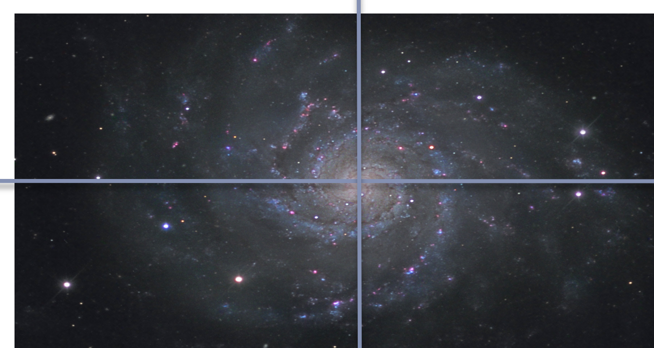
- ✓ Rotation accessible at high spectral resolution
- ✓ Vertical dispersion projected
- × Vertical height *inaccessible* except via statistical *photometric* correlations



The problem

- If you look at completely face-on galaxies you can't measure rotation ➔ can't estimate total mass (total potential)
- Even if you look at *moderate inclination* (i~30°) galaxies, you get components of the stellar velocity dispersion (σ) which are not vertical (σ_z) but radial (σ_R) or tangential (σ_ϕ).
- In other words, $\boldsymbol{\sigma}$ is a vector – the velocity ellipsoid
- From the solar neighborhood we expect: $\sigma_R > \sigma_\phi > \sigma_z$
- But we can only observe 2 spatial dimensions:
  - How do we solve for σ_z ?
- And how do we solve for σ_R, which turns out to be interesting for understanding disk heating?

1st moment: $V_{los} = V \sin i$
2nd moment: $\sigma_{los}^2 = \sigma_\phi^2 \sin^2 i + \sigma_z^2 \cos^2 i$



1st moment: $V_{los} = 0$
2nd moment: $\sigma_{los}^2 = \sigma_R^2 \sin^2 i + \sigma_z^2 \cos^2 i$

los = line of sight

Continuity Equation

- The mass of fluid in closed volume V, fixed in position and shape, bounded by surface S at time t
  - $M(t) = \int \rho(\mathbf{x},t)\, d^3\mathbf{x}$
- Mass changes with time as
  - $dM/dt = \int (\partial \rho / \partial t)\, d^3\mathbf{x} = -\int \rho\, \mathbf{v} \cdot d^2\mathbf{S}$
  - mass flowing out area-element d^2S per unit time is $\rho\, \mathbf{v} \cdot d^2\mathbf{S}$

NB: ∂ = partial derivative

- The above equality allows us to write
  - $\int (\partial \rho / \partial t) d^3\mathbf{x} + \int \rho\, \mathbf{v} \cdot d^2\mathbf{S} = 0$
  - $\int [\, \partial \rho / \partial t + \nabla \cdot (\rho\, \mathbf{v})\,]\, d^3\mathbf{x} = 0$

Divergence theorem

- Since true for any volume
  - $\boxed{\partial \rho / \partial t + \nabla \cdot (\rho\, \mathbf{v}) = 0}$

This is CE

In words: the change in density over time (1st term) is a result of a net divergence in the flow of fluid (2nd term). Stars are a collisionless fluid.

Collisionless Boltzmann Equation

- Generalize concept of spatial density ρ to phase-space density $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$, **where** $f(\mathbf{x},\mathbf{v},t)$ is the distribution function (DF)
  - $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$ gives the number of stars at a given time in a small volume $d^3\mathbf{x}$ and velocities in the range $d^3\mathbf{v}$
  - The number-density of stars at location $\mathbf{x}$ is the integral of $f(\mathbf{x},\mathbf{v},t)$ over velocities:
    - $n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$

$$\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}$$

  - The mean velocity of stars at location $\mathbf{x}$ is then given by
    - $\langle \mathbf{v}(\mathbf{x},t) \rangle = \int \mathbf{v}\, f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v} / \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$

$$\overline{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v}$$

quantities you can measure

S&G notation

Notation we'll adopt

CBE *continued*

- **Goal**: Find equation such that given $f(\mathbf{x},\mathbf{v},t_0)$ we can calculate $f(\mathbf{x},\mathbf{v},t)$ at any t, …

  …. and hence our observable quantities $n(\mathbf{x},t)$, $<v(\mathbf{x},t)>$, etc.

  - $f(\mathbf{x},\mathbf{v},t_0)$ is our initial condition
  - The gravitational potential does work on $f(\mathbf{x},\mathbf{v},t)$

- Introduce some useful notation and relate to the potential
  - Let $\mathbf{w} \equiv (\mathbf{x},\mathbf{v}) = (w_1 \ldots w_6)$
  - $\mathbf{w}' \equiv dw \,/\, dt = (\mathbf{x}',\mathbf{v}') = (\mathbf{v}, -\nabla\Phi) = (w_1 \ldots w_3, -\nabla\Phi)$

CBE *continued*

- Recall CE gives: $d\rho/dt + \nabla\bullet(\rho\,\mathbf{v}) = 0$
- Replace $\rho(\mathbf{x},t) \rightarrow f(\mathbf{x},\mathbf{v},t)$
- CE gives:
  - $df/dt + \Sigma_{i=1,6}\, d(fw_i')/dw_i' = 0$ ....but:
    - $dv_i/dx_i = 0$ x_i,v_i independent elements of phase-space
    - $dv_i'/dv_i = 0$ $v' = -\nabla\Phi$, and the gradient in the potential does not depend on velocity.
  - $df/dt + \Sigma_{i=1,6}\, w_i'(df/dw_i) = 0$
  - $df/dt + \Sigma_{i=1,3}\, [v_i(df/dx_i) - (d\Phi/dx_i)(df/dv_i)] = 0$
- $\boxed{df/dt + \mathbf{v}\bullet\nabla f - \nabla\Phi\bullet df/d\mathbf{v} = 0}$

CBE

Vector notation

Getting something useful out of CBE

- CBE is the fundamental equation of stellar dynamics
- It is a special case of Liouville's theorem:
  - the flow of particles in phase space is incompressible, i.e.
  - phase-space density is constant.
- Unfortunately, general solutions to CBE are impractical.
- However, integral moments of the CBE and velocity provide useful *dynamical* relationships between components of the velocity vector, $\mathbf{v}$, the velocity ellipsoid, $\boldsymbol{\sigma}$, and the potential, Φ.
- This will look messy (it is), but very powerful results emerge.

CBE Integrals: warm up to learn tricks

- Start by integrating CBE over all velocities (0th moment)
- $\int \{ \; (\partial f/\partial t)\, d^3\mathbf{v} + \Sigma_{i=1,3} [v_i(\partial f/\partial x_i) - (\partial \Phi / \partial x_i)(\partial f/\partial v_i)] = 0 \; \}$
  - We adopt summation convention
    - $\mathbf{A} \bullet \mathbf{B} = \Sigma_{i=1,3} A_i B_i \;\Rightarrow\; = A_i B_i$,
    - *i.e., repeated indices are implicitly summed over*

We assume the potential Φ is independent of velocity v_i

- $\int (\partial f/\partial t)\, d^3\mathbf{v} + \int v_i(\partial f/\partial x_i)\, d^3\mathbf{v} - (\partial \Phi / \partial x_i) \int (\partial f/\partial v_i)\, d^3\mathbf{v} = 0$

range of velocities does not depend on time so $\partial/\partial t$ comes outside integral and…

v_i range does not depend on x_i so $\partial/\partial x_i$ comes outside integral and…

Apply divergence theorem and the fact that $f(\mathbf{x},\mathbf{v},t)=0$ for sufficiently large $|\mathbf{v}|$, i.e., at the surface of $|\mathbf{v}| \rightarrow \infty$

0

Recall: $\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}$ and $\overline{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v}$

- $\partial \nu / \partial t + \partial(\nu \overline{v}_i)/ \partial x_i = 0$ ← *this is the continuity equation!*

Next: CBE in cylindrical coordinates

$$df\,/\,dt + \mathbf{v}\bullet\nabla f - \nabla\Phi\bullet df\,/\,d\mathbf{v} = 0$$

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0.$$

In what follows:

(1) The disk is in steady-state, so we can drop the first term

(2) we will assume the galaxy is azimuthally symmetric (e.g., a nice, circular, smooth disk) we can ignore all derivatives w.r.t. the azimuthal coordinate ϕ.

(3) The divergence theorem allows us to drop all integrals of velocity derivatives *unless* the moment is w.r.t. that velocity, in which case $v_i df/dv_i \rightarrow f$, and: $\boxed{\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}}$

CBE-v_z moment: Surface-mass density Σ_{disk}

- Multiplying CBE by v_z , integrating over all velocities, assuming steady state, azimuthal symmetry, and using the divergence theorem yields:

$$\int v_z d^3\mathbf{v} \left\{ \cancelto{0}{\frac{\partial f}{\partial t}} + v_R \frac{\partial f}{\partial R} + \cancelto{0}{\frac{v_\phi}{R}\frac{\partial f}{\partial \phi}} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right) \cancelto{0}{\frac{\partial f}{\partial v_R}} - \frac{1}{R}\left(v_R v_\phi + \cancelto{0}{\frac{\partial \Phi}{\partial \phi}} \right) \cancelto{0}{\frac{\partial f}{\partial v_\phi}} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \right\}$$

$$\frac{\partial(\nu \overline{v_R v_z})}{\partial R} + \frac{\partial(\nu \overline{v_z^2})}{\partial z} + \frac{\nu \overline{v_R v_z}}{R} + \nu \frac{\partial \Phi}{\partial z} = 0$$

CBE-v_z moment: Σ_{disk} *continued*

- 1^{st} and 3rd terms are smaller than 2^{nd} and 4^{th} by factors of $(z/R)^2$, and can be dropped.

$$\overset{\text{x}}{\frac{\partial(\nu\overline{v_R v_z})}{\partial R}} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \overset{\text{x}}{\frac{\nu\overline{v_R v_z}}{R}} + \nu\frac{\partial\Phi}{\partial z} = 0$$

- We also substitute the definition

$$\sigma_i^2 = \overline{v_i^2} - \overline{v_i}^2$$

  - Where $<v_i>$ (second term) is zero for a system in steady state

$$\boxed{\frac{\partial(\nu\sigma_z^2)}{\partial z} + \nu\frac{\partial\Phi}{\partial z} = 0} \qquad \text{(1) CBE}$$

CBE-v_z moment: Σ_{disk} *continued*

- Now use Poisson's equation to define the potential Φ in cylindrical coordinates assuming azimuthal symmetry (no dependence of ν and Φ on ϕ):

  - $4\pi G\nu(\mathbf{x}) = \nabla^2\Phi(\mathbf{x}) = d^2\Phi/dz^2 + (1/R)\, d\,[R(d\Phi/dR)]\,/\,dR$
    - Remember: $\rho = \nu_i\, m_i = <\nu><m>$; we drop <> notation here

- For $d\Phi/dR = v2(R)/R$ and $V(R)$ constant, the last term vanishes
- In general, in a highly flattened system near the mid-plane the 2nd term on the r.h.s. is much smaller than 1st term.

$$\boxed{\frac{\partial^2\Phi}{\partial z^2} = 4\pi G\,\nu} \qquad \text{(2) Poisson}$$

CBE-v_z moment: Σ_{disk} *continued*

- Next, integrate Poisson over z and relate to CBE:

$$\int_{-z}^{+z} \frac{\partial^2 \Phi}{\partial z^2}\, dz = 4\pi G \int_{-z}^{+z} \nu\, dz$$

Note $\frac{\partial \Phi}{\partial z} = 0$

at z = 0 by symmetry

$$= 2\, \frac{\partial \Phi}{\partial z}$$

←Indefinite integral
↓ Definite integral

$$\int_{-\infty}^{+\infty} \nu\, dz \equiv 4\pi G\, \Sigma_{disk}$$

Plug in to CBE

$$\boxed{\frac{\partial(\nu \sigma_z^2)}{\partial z} = -2\pi G\, \nu \int_{-z}^{+z} \nu\, dz}$$

(3) CBE+ Poisson

- To complete the calculation to find σ_z, integrate one more time in z.

CBE-v_z moment: Σ_{disk} *continued*

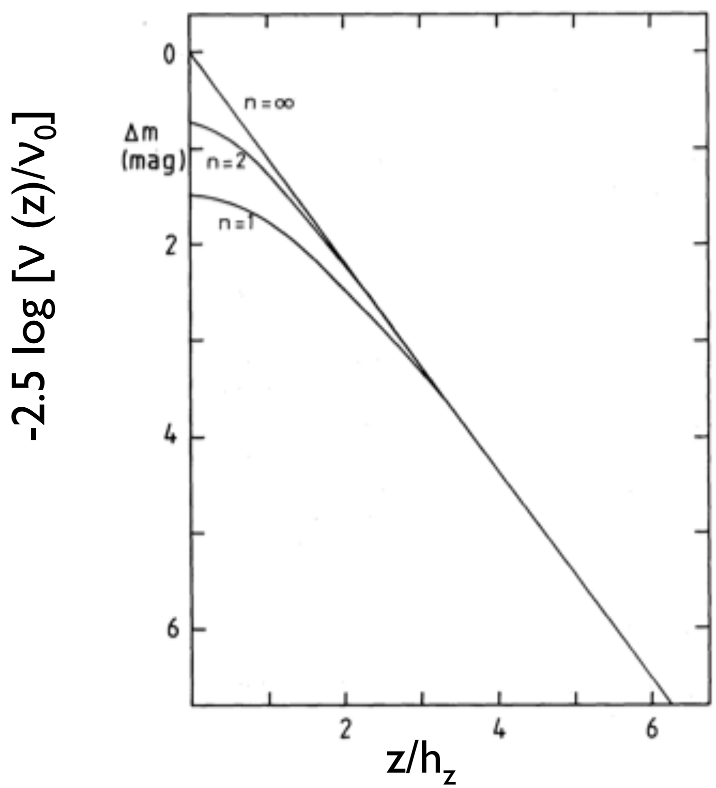
- To do this last step (integrate [3] in z), let's assume something about the mass distribution function in the vertical direction.
  - Based on what we know from light profiles of external galaxies:
    - $\nu(R,z) = \nu_0 \exp(-z/h_z - R/h_R)$
- Suggest a general vertical density function:
  - $\nu(z) = 2^{-2/n} \nu_0 \,\mathrm{sech}^{2/n}(nz/2h_z)$
    - $n=1 \rightarrow \nu(z) = (\nu_0/4)\,\mathrm{sech}^2(z/2h_z)$ *isothermal case*
    - $n=2 \rightarrow \nu(z) = (\nu_0/2)\,\mathrm{sech}(z/h_z)$ *intermediate*
    - $n=\infty \rightarrow \nu(z) = \nu_0 \exp(z/h_z)$ *what's observed (maybe)*
- The surface-density Σ_{disk} follows from direct integration:
    - $n=1 \rightarrow \Sigma_{disk} = \nu_0 h_z$
    - $n=2 \rightarrow \Sigma_{disk} = (\pi/2)\, \nu_0 h_z$
    - $n=\infty \rightarrow \Sigma_{disk} = 2\nu_0 h_z$

CBE-v_z moment: Σ_{disk} *continued*

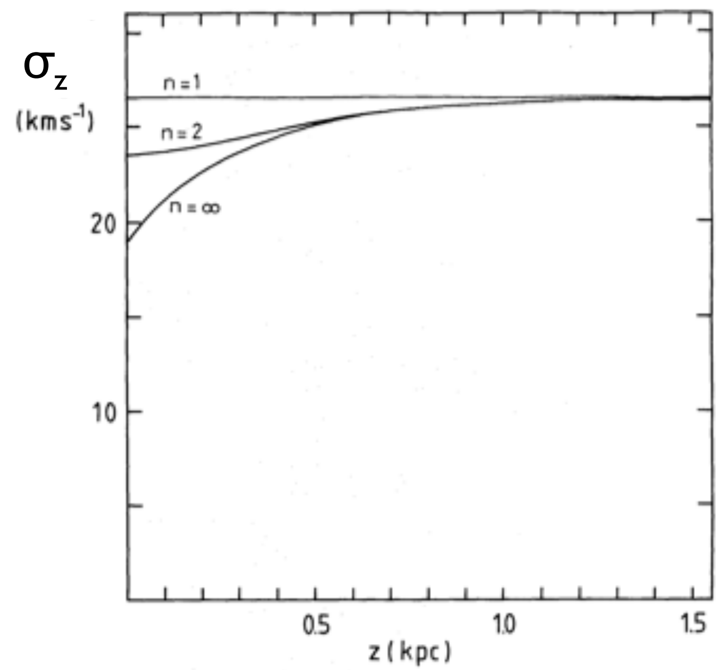
- The gradient of the potential follows from the corresponding indefinite integral:
  - $\frac{\partial \Phi}{\partial z} = 2\pi G \int \nu \, dz$
    - $= 2\pi G \nu_0 h_z \tanh(z/2h_z)$, $n = 1$
    - $= 2\pi G \nu_0 h_z \arctan[\sinh(z/h_z)]$, $n = 2$
    - $= 2\pi G \nu_0 h_z [1 - \exp(z/h_z)]$, $n = 3$
- Lastly, we integrate the gradient of the potential $\frac{\partial \Phi}{\partial z}$ and divide by ν to solve for σ_z^2:
  - $\sigma_z^2 = 2\pi G \, h_z \, \Sigma_{disk}$ $n = 1$
  - $\sigma_z^2 = 1.7051 \, \pi \, G \, h_z \, \Sigma_{disk}$ $n = 2$
  - $\sigma_z^2 = 3\pi/2 \, G \, h_z \, \Sigma_{disk}$ $n = 3$

CBE-v_z moment: Σ_{disk} *continued*

- If the disk is locally isothermal, $d\sigma_z^2/dz = 0$
  - Why is this? What does isothermal mean in terms of kinematic motion?



$\Sigma_{disk} = 80\ M_\odot\ pc^{-2}$, h_z =325 pc



van der Kruit 1988, A&A, 192, 117

Finally….the Disk Mass formula

Use *statistical* measure of *disk thickness* from edge-on galaxies …

vertical distribution*

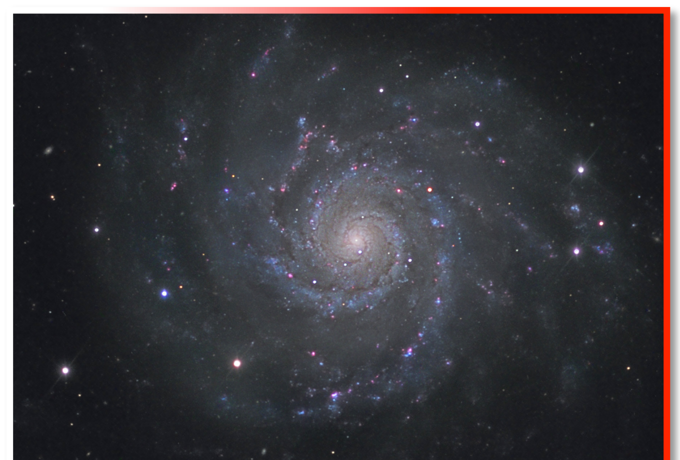
thickness

vertical oscillations

$$\Sigma = 100 \left(\frac{k}{3/2} \right)^{-1} \left(\frac{h_z}{444\ \mathrm{pc}} \right)^{-1} \left(\frac{\sigma_z}{30\,\mathrm{km/s}} \right)^{2} \mathrm{M_{sol}\ pc^{-2}}$$

Disk mass surface density

….and apply relation to face-on galaxies where the *vertical oscillations* of stars can be measured.



* 1.5<k<2 for exp, sech, sech^2

CBE-v_R and $v_R v_\phi$ moments:

- Multiplying CBE by $v_R v_\phi$, integrating over all velocities, assuming steady state, azimuthal symmetry, and using the divergence theorem yields:

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z v_\phi})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v_\phi} R\frac{\partial\Phi}{\partial R} - 2\overline{v_R^2 v_\phi}\right) = 0$$

- Multiplying CBE by v_R, integrating over all velocities, and assuming azimuthal symmetry (ϕ-derivatives=0) yields:

$$\frac{\partial(\nu\overline{v_R})}{\partial t} + \frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial(\nu\overline{v_z v_R})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) = 0$$

CBE-v_R and $v_R v_\phi$ moments: Epicycle approximation

- The CBE-v_R and $v_R v_\phi$ moments combined with this identify (valid when ellipsoid is aligned with the potential andsymmetric about v_ϕ):

$$\overline{(v_\phi - \overline{v_\phi})^3} = (\overline{v_\phi^3} - \overline{v_\phi}\,\overline{v_\phi^2}) - 2\overline{v_\phi}(\overline{v_\phi^2} - \overline{v_\phi}^2) = 0$$

yield
$$\overline{v_R^2}\left(\frac{\partial \overline{v_\phi}}{\partial R} + \frac{\overline{v_\phi}}{R}\right) - \frac{2\overline{v_\phi}}{R}(\overline{v_\phi^2} - \overline{v_\phi}^2) = 0$$

- Which can be rearranged to give:

$$\boxed{\frac{\sigma_\phi^2}{\sigma_R^2} = \frac{1}{2}\left(\frac{\partial \ln \overline{v_\phi}}{\partial \ln R} + 1\right)}$$

This is powerful because it gives us another piece of information to uncover all of the ellipsoid components $\sigma_R : \sigma_\phi : \sigma_z$

CBE-v_R moment: Asymmetric drift

- Eliminating time derivatives and assuming there are no streaming motions ($<v_r>^2=0$) yields:

$$v_c^2 - \overline{v_\phi}^2 = \sigma_\phi^2 - \sigma_R^2 - \frac{R}{\nu}\frac{\partial(\nu\sigma_R^2)}{\partial R} - R\frac{\partial(\overline{v_r v_z})}{\partial z}$$

  - *Collisionless particles have tangential velocities smaller than the circular speed of the potential, in quadrature proportion (**think: energy**) to their velocity dispersion.*
  - This is powerful because it relates the velocity dispersion ellipsoid components to tangential velocities, thereby giving us another piece of information to uncover all of the ellipsoid components $\sigma_R : \sigma_\phi : \sigma_z$

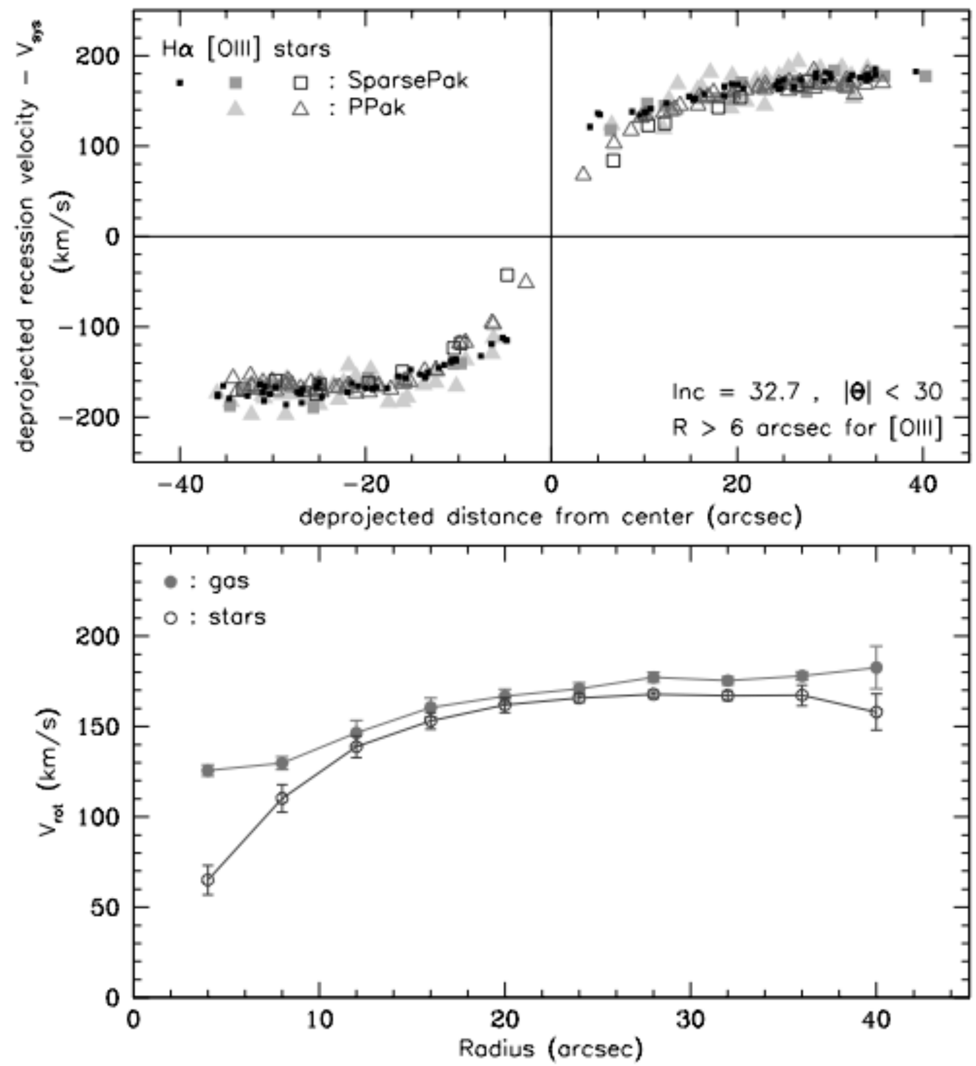
- Now the problem is *over* constrained, i.e., σ_{maj}, σ_{min} plus *two* dynamical relations (epicycle approx. and asymmetric drift).
  - A *good* thing because there are a lot of assumptions.

Asymmetric drift

- Assume the gas tangential velocity is close to v_c
  - Why is this reasonable?
- V_ϕ is the tangential velocity of the stars

Bershady et al. 2010

UGC 6918



Wrapping up:

- If we make some assumptions
  - about the distribution function $\nu(R,z)$, namely a double exponential in R and z,
  - that the ellipsoid tilt yields a last term between 0 and σ_z^2
  - and we substitute in the epicycle approximation to eliminate σ_ϕ

$$v_c^2 - \overline{v_\phi}^2 \approx \sigma_R^2 \left(\frac{1}{2} \frac{\partial \ln \overline{v_\phi}}{\partial \ln R} + \frac{2R}{h_\sigma} + \frac{R}{h_R} - 1 \right) + \frac{\sigma_z^2}{2}$$

- This formula, plus direct measurments of
  - v_c, v_ϕ, σ_{maj}, σ_{min}

  are our best-bet combination for
  - directly measuring Σ_{disk}
  - decomposing rotation curves,
  - determing disk M/L, and
  - the dark-mater density distribution.