ASTRO 735

Supplementary Notes #3
Exotic Cosmological Models

1 Introduction

These notes summarize the necessary equations to calculate observable quantities (time, apparent size, apparent brightness, and volume) in exotic cosmologies. 'Exotic' is defined when the energy density today has a component other than purely non-relativistic matter. The focus here is on $\Lambda \neq 0$, particularly the special cases when space is flat ($k = 0$). But the notes are general enough for you to do any calculation, for any k, as long as you can specify the dependency of the various forms of energy density on the scale factor a. Most of these notes are lifted directly from Carroll, Press, & Turner (1992, ARAA, 30, 499) and Charlton & Turner (1987, ApJ, 313, 495), with some minor corrections for typographical errors and consolidation of the nomenclature to fit my personal taste and to be consistent with what I have used earlier in the course. (I try to point out these differences to avoid confusion.) The Charlton & Turner (1987) paper is very nice for the special case where $k = 0$, and fine in general for look-back time. However, their formulation for distance (and hence volume) is only valid for $k = 0$, as they note.

The key starting point is that all critical observables in a Friedmann-Robertson-Walker (FRW) cosmology are determined by integral functions of the scale factor $a(t)$, or a for short [$a_0 = a(t_0)$, i.e. today] and the time derivative $\dot{a}$. These can be completely specified by manipulation of only one of the two Friedmann equations. Recall that by definition: $1 + z \equiv a_0/a$ and $H \equiv \dot{a}/a$. In what follows, first I lay out how to derive look-back times, distances and volumes in terms of a, $\dot{a}$, and then I explore the relevant 'Friedmann equation' in general form from which $\dot{a}/a$ can be determined. If you understand these notes, calculating observable in any FRW cosmology will be forever trivial.

2 Look-back time

The general expression for the age of the universe at some epoch z is:

$$t(z) = \int_0^{a(t)} da/\dot{a} \tag{1}$$

which follows directly from the definition of $\dot{a}$. The look-back time from now to a past event at epoch z is simply $t_0 - t(z)$:

$$t_0 - t(z) = \int_{a_0/(1+z)}^{a_0} da/\dot{a} \tag{2}$$

Using the Friedman equation for $\dot{a}/a$, this integral can be solved analytically or numerically in straight-forward fashion via substitution for $\dot{a}$ as a function of a/a_0, Ω_0, and H_0. (In the the general case Ω_0 is a series of terms associated with different factors of a/a_0, as we show in §5 below.) This is Charlton & Turner's (1987) approach, which is the most direct. With more algebra you can derive Carroll *et al.* 's formula which is an integral over z' from 0 to z. This formula is useful for numerical integration but is not as general as equation (2), since Carroll *et al.* only consider non-relativistic matter and non-zero Λ.

3 Distances and volumes in a flat universe: $k = 0$

For a flat space-like universe, we can define a useful quantity, which we call the 'proper distance' between an observer and some object at redshift z:

$$d_0(z) = c\ a_0 \int_{t_0-t}^{t_0} dt'/a(t') \tag{3}$$

The proper distance is the present value of the scale factor times an integral representing the radial coordinate distance covered by a photon from redshift z until today. In previous notes we have referred to this radial coordinate distance as r; in Charlton & Turner (1987) and Carroll *et al.* (1992) the nomenclature for the same quantity is r_1. (c is the speed of light, which, unfortunately, is often set to 1 in the literature, and hence hidden; here we include it explicitly.) To make this integral formula useful, we do a variable substitution combining a and $\dot{a}$ for dt to get:

$$d_0(z) = c\ a_0 \int_{a_0/(1+z)}^{a_0} da/(a\dot{a}) \tag{4}$$

which, again, is in a convenient form to use with the Friedman equation. With the proper distance in hand, all else follows, and the Universe is yours (here, for flat space only)!

The luminosity distance:

$$d_l(z) = d_0(z)(1+z) \tag{5}$$

such that $l/L = 1/4\pi d_l(z)^2$, where l is the apparent brightness and L is the luminosity.

The angular size distance:

$$d_A(z) = d_0(z)(1+z)^{-1} \tag{6}$$

such that $\theta(z)/D = d_A(z)^{-1}$, where $\theta(z)$ is the apparent size in radians at redshift z of some source of physical size D.

The differential volume element over 4π steradians:

$$dV(z) = \frac{4\pi}{3}\frac{d}{dz}[d_0(z)^3]\ dz \tag{7}$$

which is proportional to the number of sources one would count over the whole sky in some redshift (volume) shell between z and $z + dz$ for a source population which does not evolve in comoving number density. The derivative works out such that (try it, it's easy):

$$dV(z) = cH(z)^{-1} d_l(z)^2 (1+z)^{-2}\ dz. \tag{8}$$

Note that this formula has a direct dependence on $H(z)^{-1} = a/\dot{a}$ in addition to factors of $d_l(z)$ and kinematics factors of $(1+z)$.

Also note that all distances ultimately scale as c/H_0, and dV/dz as $(c/H_0)^3$; c/H_0 is the only dimension in town. Once we can write down an expression for $\dot{a}$ as a function of a, we're done; it is just a question of algebraic manipulation and possibly some numerical integration to get $d_0(z)$. Again, the expression for $\dot{a}$ will come from one of the Friedmann equations.

BUT!!! while this formulation is relatively simple and intuitive (at least in the case for volume), it is only valid for $k = 0$, flat space-like universe. It is because $k = 0$ that the math is simple; it is because space is flat that volume formulation is intuitive. The good news is that you *can* use this approach for the take-home final for the non-zero Λ cases.

4 Distances and volumes: any k

Here's what happens when k isn't necessarily 0. This follows directly from Carroll *et al.* (1992), with a few more steps added in. The fundamental formulations are that

$$d_l(z) = (a_0/a)\ a_0\ r \tag{9}$$

$$d_A(z) = (a_0/a)^{-1}\ a_0\ r \tag{10}$$

$$dV = 4\pi\ a_0^3\ \frac{r^2}{(1-kr^2)^{1/2}}\ dr \tag{11}$$

where as before r is the radial coordinate of the metric corresponding to the position of some observed source at some distance from the observer. The volume element is, again, for 4π steradians.

If you replace $(a_0/a) = (1+z)$, then the distance formulae should look familiar. In the earlier supplemental notes we motivated these extra factors of $(1+z)$ in terms of time dilation and energy diminution (for $d_l(z)$) and angular aberration (for $d_A(z)$). The differential volume formula should look unfamiliar because here we have written it in general terms. You can use the definition of $d_l(z)$ to put this in a slightly more familiar form, but note that it cannot be reduced to equation (7) or the formula from 'Supplemental Notes #2' because these were special cases for k or the energy terms in the Friedmann equation. The above equation is completely general for any FRW cosmology.

The key to using these formulae is to solve for r in terms of a (i.e. redshift). We do this by considering the null geodesic (ray-tracing a photon over the past light-cone):

$$\frac{1}{c}\frac{dr}{dt} = \frac{(1-kr^2)^{1/2}}{a} \tag{12}$$

(To derive this, simply go back to the definition of the metric, and set it equal to zero.) To get something useful, re-arrange terms and integrate both sides:

$$\int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = c\int_{t_0-t}^{t0} dt'/a(t') \tag{13}$$

and then make the same substitution going from equation (2) to (3):

$$\int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = c\int_{a_0/(1+z)}^{a_0} da/(a\dot{a}) \tag{14}$$

The left hand side of (13) has an analytic solution for all cases $k = -1, 0, +1$. The solution for $k = 0$ is in the previous section, i.e. $d_0(z) = a_0\ r$, as can be immediately seen. The right hand side of (13) has analytic solutions for special cases. The general procedure is simple, however. For each of three values of k, r is a unique function of the right hand of (13), which can be solved either analytically or numerically.

Carroll *et al.* (1992) write down the integral for the general case where Λ is non-zero. This is straight-forward (but tedious) to derive in their form using the Friedmann equation (15) below. There formula is not, however, a completely general solution for all types of energy densities, whereas (14), as stated is completely general within the context of FRW.

5 The Friedmann equations

The Friedmann equation for $(\dot{a}/a)^2$, which can be thought of loosely as an 'energy balance' equation, can be written in its most general form as:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{c^2\Lambda}{3} - \frac{c^2 k}{a^2} \tag{15}$$

where Λ is the cosmological constant[1] and $\rho = \sum_i \rho_i$ is the sum of energy density from all forms of matter and other exotica (e.g. cosmic strings, etc., see Charlton & Turner for some examples). A key point is that different types of energy density will have different dependencies on a. For example, ρ for non-relativistic and relativistic matter (and radiation) scale as a^{-3} and a^{-4}, respectively. The curvature-term scales as a^{-2}, and the Λ-term is constant (as you might expect!). It is useful to re-write (15) in terms of contributions in the form of Ω. We define:

$$\Omega_i \equiv \frac{8\pi G\rho_i}{3H^2} \tag{16}$$

$$\Omega_\Lambda \equiv \frac{c^2\Lambda}{3H^2} \tag{17}$$

[1]In Impey's notes and elsewhere it is more common to define $\Lambda' = c^2\Lambda$, but this is just definitioinal since both are constant.

$$\Omega_k \equiv \frac{-c^2 k}{H^2 a^2} \tag{18}$$

where ρ_i is the energy density of particles, radiation, strings, etc. These are general expressions for all epochs. Values today, denoted $\Omega_{i,0}$, $\Omega_{\Lambda,0}$, and $\Omega_{k,0}$, respectively, are found by replacing ρ_i with $\rho_{i,0}$, H with H_0, and a with a_0. As an example, consider only two forms for ρ_i: non-relativistic matter (Ω_M) and radiation and relativistic matter (Ω_R). The Friedmann equation (15) can then be written as:

$$\left(\frac{H}{H_0}\right)^2 = \frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{R,0}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}\left(\frac{a}{a_0}\right)^{-2} \tag{19}$$

This equation can be used to solve all of the integrals in the previous section for any case you care to consider. Want another type of energy density? Just figure out how it scales with a/a_0 and add a new Ω-term into the equation.

It is instructive to note that (19) implies (by setting $t = t_0$):

$$1 = \sum_i \Omega_{i,0} + \Omega_{\Lambda,0} + \Omega_{k,0} \tag{20}$$

What we conventionally think of as Ω we can define as $\Omega_{tot} \equiv \sum_i \Omega_i + \Omega_\Lambda = 1 - \Omega_k$. It is the value of Ω_{tot} that determines whether the universe is open, closed, or critical, which is readily evident from its dependence on Ω_k.

Finally, let's not forget $q \equiv -(a\ddot{a}/\dot{a}^2)$. We need the second Friedmann equation for this, namely:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\sum_i \rho_i + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3} \tag{21}$$

In the general case, q is simply minus the ratio of (21) to (15). For $t = t_0$, we still have fairly generally:

$$q_0 = \frac{1}{2}\sum_i \Omega_{i,0} - \Omega_{\Lambda,0} \tag{22}$$

where the only assumption we have made is that $\sum_i \rho_{i,0} c^2 >> p_0$, which one hopes is a pretty safe bet.