Astronomy 330

Lecture 10

06 Oct 2010

Outline

- Review Galactic dynamics
  - Potentials
  - Energetics
- Rotation curves
  - Disk-halo degeneracy
- Characteristics of dynamical systems
- Dynamics of collisionless systems

But first ….

…..this is part of why you went to college and became an astronomy major, right?

Go check it out and report back on Friday.

Review: Energy considerations

- Recall:
  - Newton: $d(mv)/dt = - m \nabla \Phi(\mathbf{x})$,
  - Poisson: $\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x})$
  - Total energy defined: $E = KE + PE = ½ mv^2 + m \Phi(\mathbf{x})$
  - $PE = W = ½ \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}$
  - Escape velocity (E=0): $v_e(\mathbf{x}) = (2|\Phi(\mathbf{x})|)^{1/2}$
- Virial Theorem: $KE = -½ PE$ for systems in equilibrium
  - In practice: this is the way we measure masses for astronomical objects because KE measured independent of potential and PE is proportional to M.

Virial Theorem

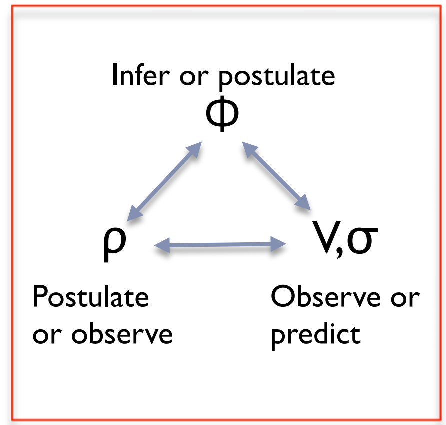
- Why is KE = −½ PE for equilibrium?
- Define moment of inertia
  - $I = m\,\mathbf{x}\cdot\mathbf{x}$
- Then it follows:
  - $\frac{1}{2}\, d^2(I)/dt^2 = m\, d/dt\,(\,\mathbf{x}\cdot d\mathbf{x}/dt\,)$
  - $= m\, d/dt\,(\mathbf{x}\cdot\mathbf{v})$
  - $= m\,[\, d\mathbf{x}/dt\cdot\mathbf{v} + \mathbf{x}\cdot d\mathbf{v}/dt\,]$
  - $= m\, v\cdot v + d/dt(m\mathbf{v})\cdot\mathbf{x}$

↑ $2\,KE = mv^2$ ↑ $-m\nabla\Phi(\mathbf{x})\cdot\mathbf{x} = PE$

- For a system in equilibrium: $d^2(I)/dt^2 = 0$
  - I is time-independent
  - Is dI/dt interesting?

Review: Application of potentials to galaxies

- Here's the process:
  - We start by looking at some very simple geometric cases
  - Define a few terms that help us think about and characterize the potentials
  - Become more sophisticated in the form of the potential to be more realistic in matching galaxies
- Concepts:
  - circular and escape velocities
  - Time scales: dynamical, free-fall
  - Potential (W or PE) and kinematic energy (K or KE)
  - Energy Conservation and Virial Theorem
  - Angular momentum
- Example: rotation curves of galaxies



Spherical distributions: characteristic velocities

- The gravitational attraction of a density distribution, $\rho(r')$, on a particle at distance, r, is:
  - $F(r) = -m(d\Phi/dr) = -GmM(r)/r^2$
  - $M(r) = 4\pi\int\rho(r')r'^2dr'$
- Circular speed:
  - In any potential $d\Phi/dr$ is the radial acceleration
  - For a circular orbit, the acceleration is v^2/r
  - ➔ $v_c^2 = r(d\Phi/dr) = GM(r)/r$
    - outside a spherical mass distribution, v_c goes as $r^{-1/2}$
      - Keplerian
- Escape speed: $v_e(r) = (2|\Phi(r)|)^{1/2} = [2\int GM(r)dr/r^2]^{1/2}$

Homogeneous Sphere: characteristic time-scales

- $M(r) = (4/3)\pi r^3 \rho$
  - ρ is constant
- For particle on circular orbit, $v_c = (4\pi G \rho /3)^{1/2} r$
  - rises linearly with r.
  - Check out the Galaxy's inner rotation curve.
  - What does this say about the bulge?
- Orbital period: $T = 2\pi r/v_c = (3\pi/G \rho)^{1/2}$
- Now release a point mass from rest at r:
  - $d^2r/dt^2 = -GM(r)/r^2 = -(4\pi G \rho /3)r$
  - Looks like the eqn of motion of a harmonic oscillator with frequency $= 2\pi/T$
  - Particle will reach r = 0 in 1/4 period (T/4), or
  - $t_{dyn} \equiv (3\pi /16G \rho)^{1/2}$

Isochrone Potential

- *Since nothing is really homogeneous…*
- $\Phi(r) = -GM/[b+(b^2+ r^2)^{1/2}]$
  - b is some constant to set the scale
  - $v_c^2(r) = GMr^2/[(b+a)^2a]$ ➔ $(GM/r)^{½}$ at large r
  - $a \equiv (b^2+r^2)^{½}$
- This simple potential has the advantage of having constant density at small r, falling to zero at large r
  - $\rho_0 = 3M / 16\pi Gb^3$
- Similar to the so-called Plummer model used by Plummer (1911) to fit the density distribution of globular clusters:
  - $\Phi(r) = -GM / (b^2+ r^2)^{1/2}$
  - $\rho(r) = (3M / 4\pi Gb^3)(1+r^2/b^2)^{-5/2}$

Singular Isothermal Sphere

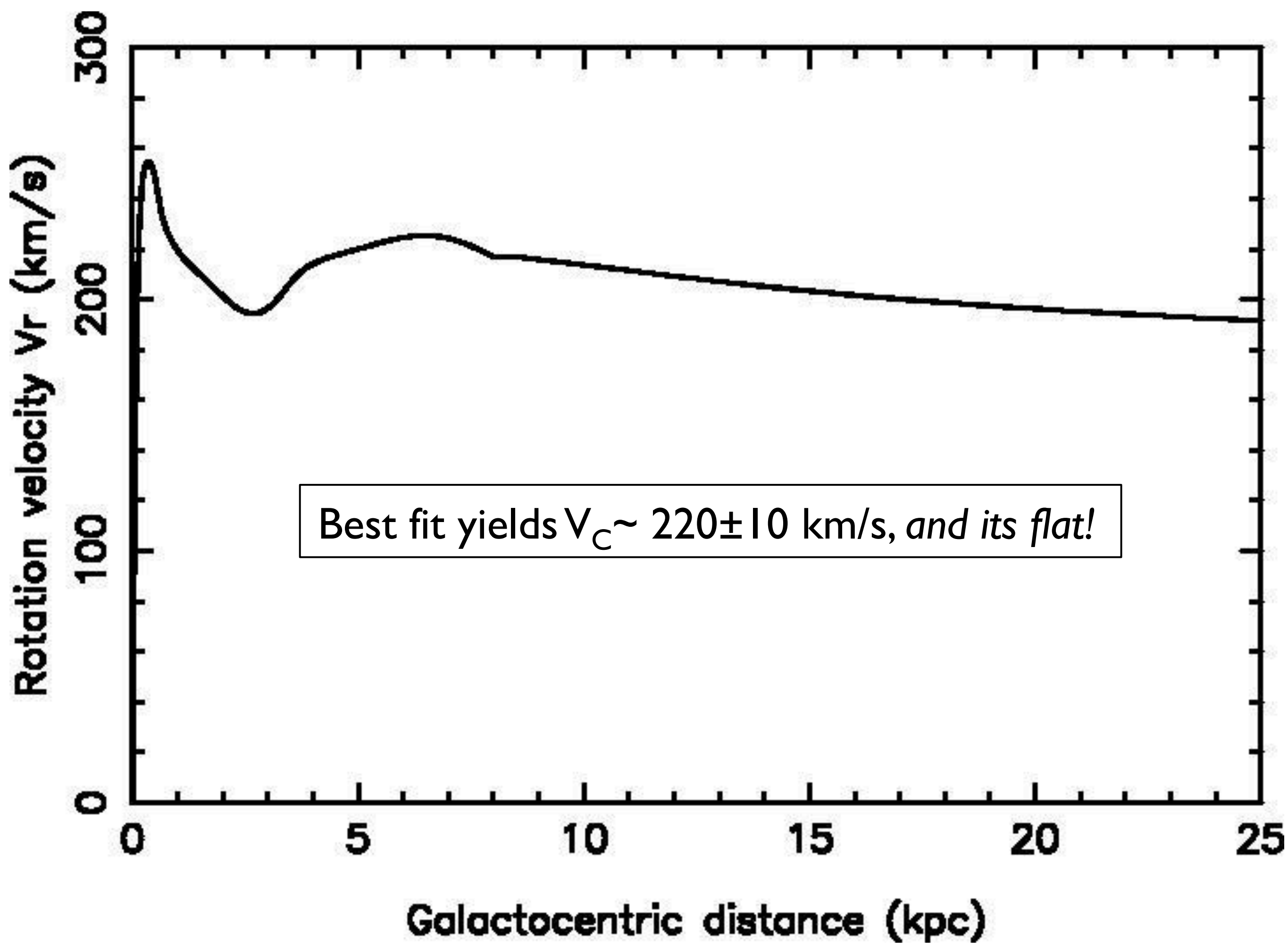
- Physical motivation:
  - Hydrostatic equilibrium: pressure support balances gravitational potential
  - $dp/dr = (k_B\,T/m)\,d\rho/dr = -\rho\,GM(r)/r^2$
    - $\rho(r) = \sigma^2/2\pi G r^2$
      - where $\sigma^2 = k_B\,T/m$
- Singular at origin so define characteristic values:
  - $\rho' = \rho/\rho_0$
  - $r' = r/r_0$
  - $r_0 \equiv (9\sigma^2 / 4\pi G\rho_0)^{1/2}$
- $\Phi(r)$ is straight-forward to derive given our definitions:
  - $\Phi(r) = V_c^2 \ln(r/r_0)$
  - $v_c = 4\pi\rho_0\, r_0^2$
- A special class of power-law potentials for α=2
  - $\rho(r) = \rho_0\,(r_0/r)^\alpha$
  - $M(r) = 4\pi\rho_0\, r_0^\alpha\, r^{(3-\alpha)}/\ (3-\alpha)$
  - $v_c^2(r) = 4\pi\rho_0\, r_0^\alpha\, r^{(2-\alpha)}/\ (3-\alpha)$

Look what happens to V(r) when α=2

Pseudo-Isothermal Sphere

- Physical motivation: avoid singularity at r=0, but stay close to functional form. Posit:
  - $\rho(r) = \rho_0[1 + (r/r_c)^2]^{-1}$
- $\Phi(r)$ is straight-forward to derive given our definitions
- $V(r) = (4\pi G \rho_0 r_c^2 [1-(r_c/r)\arctan(r/r_c)])^{1/2}$
  - This gives a good match to most rotation curves within the optical portion of the disk.
  - But it does not give a good description of the light distribution of disks.

Flat rotation curves: the Milky Way



Nakanishi & Sofue (2003, PASJ, 55,191)

Flat rotation curves: external galaxies

Which looks most like the MW?
Why different shapes and extents?

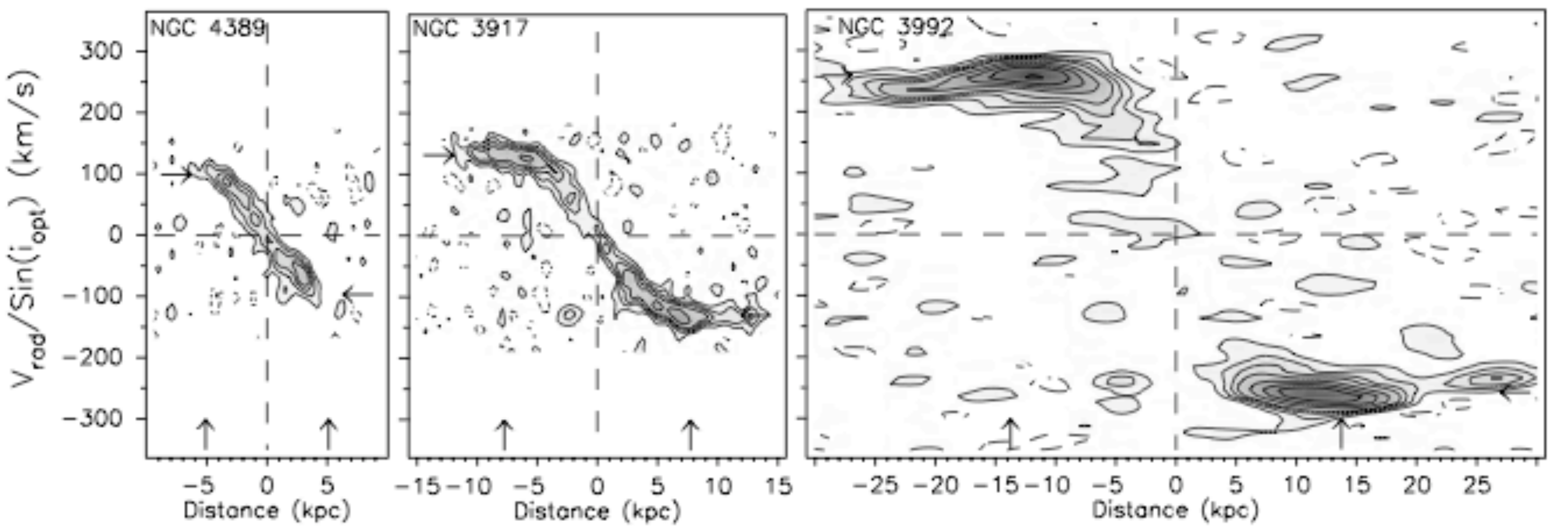


FIG. 2.—Prototype examples of the three categories of rotation curves. *Left:* Galaxy with a rotation curve that rises continuously until the last measured point. The measured maximum rotational velocity V_{max} is set by the extent of the H I disk (*R* curve). *Middle:* "Classical" rotation curve; a gentle rise in the central regions with a smooth transition into the extended flat part (*F* curve). *Right:* Rotation curve that reaches a maximum in the optical regions after which it declines somewhat to an extended flat part in the outer disk. In this case, the maximum rotation velocity exceeds the amplitude of the flat part (*D* curve). The vertical arrows indicate $\pm R_{25}$, and the horizontal arrows indicate the rotational velocities as inferred from the global profiles.

Verheijen 2005, ApJ, 563, 694

Flat rotation curves: external galaxies

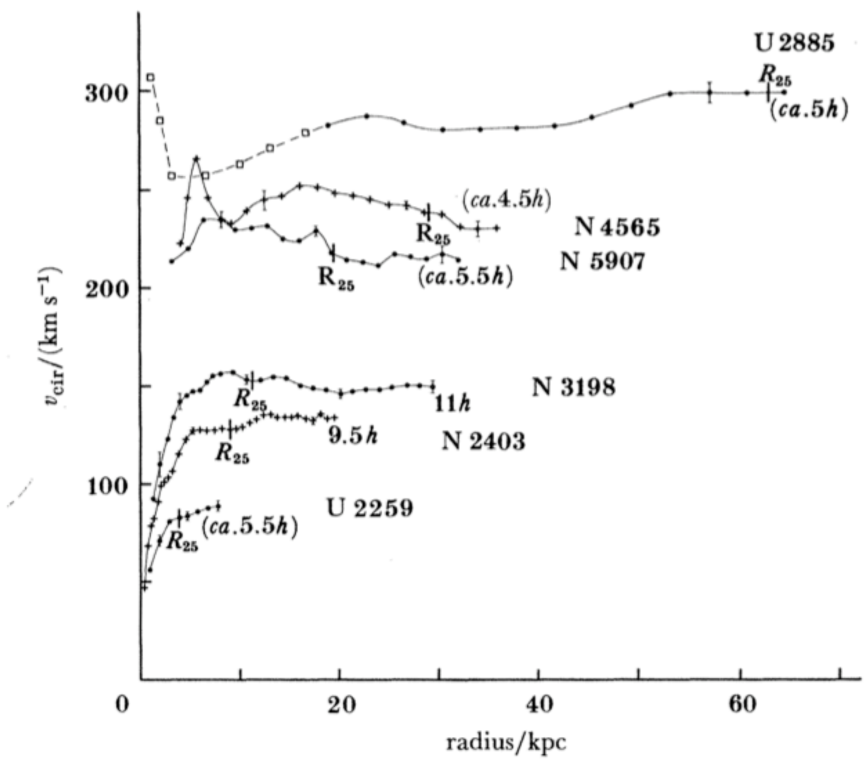


FIGURE 2. HI rotation curves for a number of spiral galaxies (Sancisi & van Albada 1986). Distances are based on $H_0 = 75$ km s^{-1} Mpc^{-1}. The optical radius, R_{25}, and the number of disc scalelengths, h, at the last measured point are indicated. For the inner region of UGC 2885 optical velocities (Rubin *et al.* 1986) have been used. All curves remain approximately flat beyond the turnover radius of the disc (2.5 h).

Van Albada et al. 1986, *Phil. Trans. Royal Soc. London*, 320, 1556, 447

Flat rotation curves: the disk

- Disk component
- $\Sigma(r) = \Upsilon \times \mathrm{dexp}\{-0.4[\mu(r)-\mu_0]\}$
  - Σ is the mass surface-density
  - Υ is the mass–to–light ratio (M/L)
  - μ is the surface–brightness (mag arcsec^{-2})
  - Surface mass density ($M_{\odot}$ pc^{-2}) is just the mass-to-light ratio times the surface brightness (converted to $L_{\odot}$ pc^{-2})
- Mass ➔ potential ➔ circular velocity
  - The trick here is to deal with the non-spherical density distibution.

Flat rotation curves: the exponential disk

- $\Sigma(r) = \Sigma_0 \exp(-r/h_R)$
- Mass:
  - $M(r) = 2\pi\int \Sigma(r')r'\,dr' = 2\pi \Sigma_0 h_R^2[1-\exp(-r/h_R)(1+r/h_R)]$
- ➔ potential
  - $\Phi(r,z=0) = -\pi G \Sigma_0\, r\, [I_0(y)K_0(y) - I_1(y)K_1(y)]$
    - $y = r/2h_R$
    - I, K are modified Bessel functions of the 1st and 2nd kinds.
- ➔ circular velocity
  - $V_c^2(r) = r\, d\Phi/dr = 4\pi G \Sigma_0\, h_R\, y^2\, [I_0(y)K_0(y) - I_1(y)K_1(y)]$

A bit of work; see Freeman (1970) and Toomre (1963)

- Note: This is for an infinitely-thin exponential disk. In reality, disks have a thickness [$\rho(r,z) = \rho_0 \exp(-r/h_R) \exp(-z/h_z)$] with axis ratios h_R:h_z between 5:1 and 10:1

Rotation from a double-exponential disk

$\rho(r,z) = \rho_0 \exp(-r/h_R) \exp(-z/h_z)$

Disk oblateness:

$q \equiv h_z/h_r$

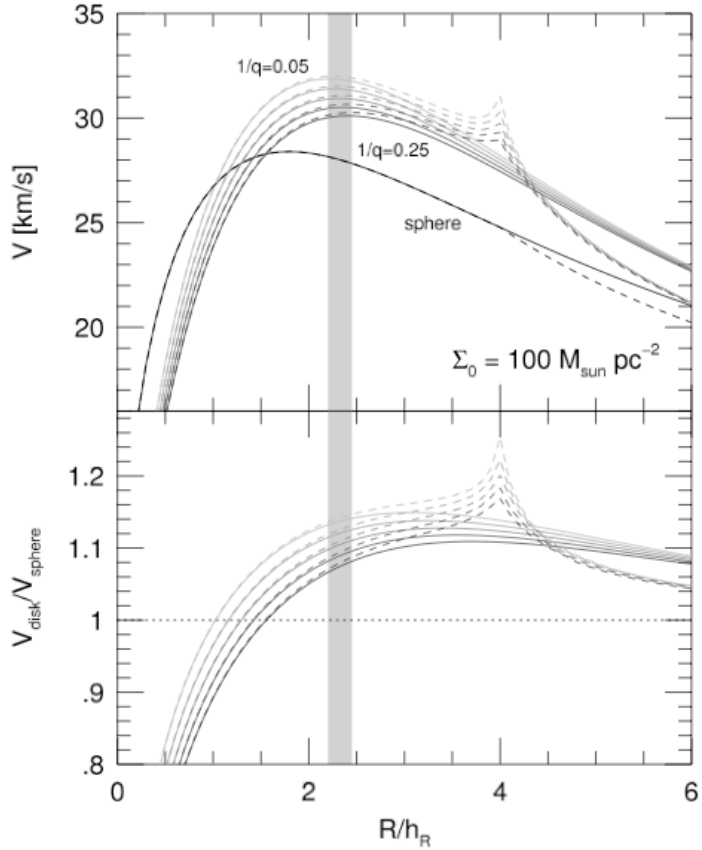
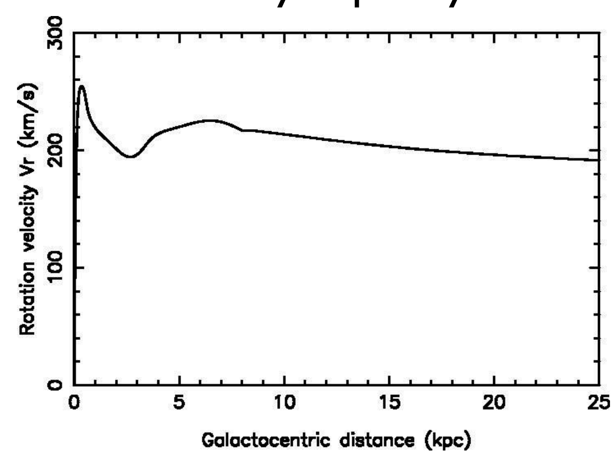


Fig. 17.— Rotation speed of an exponential disk with central mass surface density of 100 $\mathcal{M}_\odot$ pc^{-2} and oblateness $0.05 < q < 0.25$ versus radius normalized by scale-length, compared to a spherical density distribution with the same enclosed mass. Bottom panel shows the ratio of spherical to disk velocities. Dashed and solid lines show disks truncated at R/h_R=4 and 10, respectively. The radial range where these disks have peak velocities is shaded in gray.

← A pure disk isn't flat

↓ Reality is pretty flat



→ So what makes the observed rotaion curve of galaxies flat?

Flat rotation curves: the halo

- "Halo" component – we need V(r) to be constant at large radius (the bulge helps only at small r).
- One option is the singular isothermal sphere, here V(r) is constant at all radii.
  - Is that plausible given observed rotation curves (e.g, MW)?
- Another option: the pseudo-isothermal sphere
  - $\rho(r) = \rho_0[1 + (r/r_c)^2]^{-1}$
  - $V(r) = (4\pi G \rho_0 r_c^2 [1-(r_c/r)\arctan(r/r_c)])^{1/2}$
  - This gives a good match to most rotation curves within the optical portion of the disk.
- Also the NFW* profile, motivated by cold-dark-matter (CDM) structure-formation simulation (see S&G p.117):
  - $\rho_{NFW}(r) = \rho_n(r/a_n)^{-1}[1 + (r/a_n)]^{-2}$
  - $V_{NFW}(r) = (4\pi G \rho_n a_n^2 [\ln(1 + (r/a_n))/(r/a_n) - 1/(1 + (r/a_n))])^{1/2}$

*Navarro, Frenk & White 1996

The Disk-Halo Degeneracy

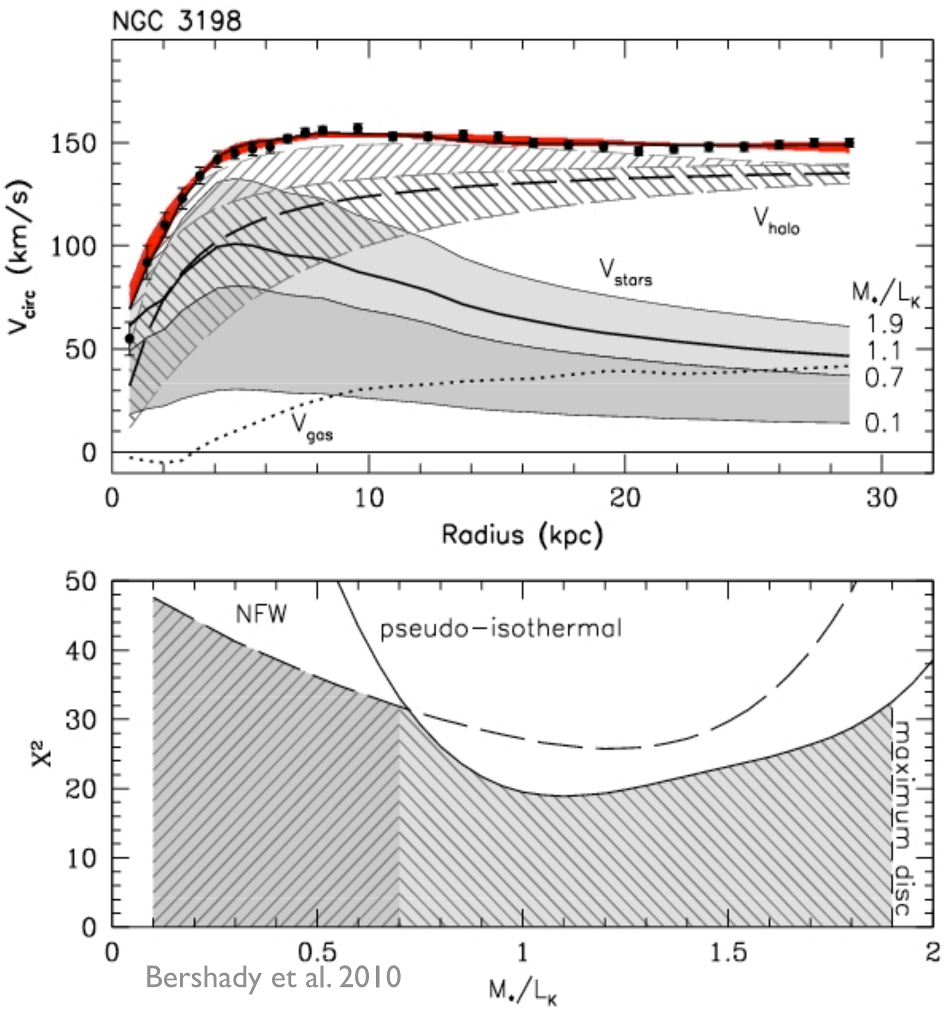
- **Q:** Is it possible to decompose the rotation curve of a spiral galaxy into disk, bulge, and halo components?
- Mass decomposition:
  - Recall $v_c^2 = r(d\Phi/dr)$
  - $\Phi = \Sigma_i \Phi_i$, i = bulge, disk, halo, kitchen sink
  - For spherical mass distribution
    - $r(d\Phi/dr) = GM(r)/r$
  - For a flattened mass distribution define f_i such that
    - $r(d\Phi/dr) = f_i GM(r)/r$
    - $v_c^2 = \Sigma_i f_i GM_i(r)/r = \Sigma_i v_{c,i}^2$
  - Measure v_c^2
  - Estimate individual components $v_{c,i}^2$ constrained by $v_c^2 = \Sigma_i v_{c,i}^2$
- Can it be done with any reasonable fidelity?

The Disk-Halo Degeneracy

- **Q:** Is it possible to decompose the rotation curve of a spiral galaxy into disk, bulge, and halo components?
- **A:** *No; Solutions are degenerate*
- Degeneracies:
  - Unconstrained fitting functions for halo:
    - e.g., pseudo-isotherm. vs NFW
  - Disk M/L (Υ_{disk}) uncertain
    - Stellar populations Υ_*: depends on SFH, IMF, and detailed knowledge of all phases of stellar evolution.
    - ISM
      - Gas
        - Atomic: straightforward to measure
        - Molecular: harder to measure
      - Dust: probably insignificant
    - Dark matter?
  - Non-circular motions
- However, it is possible to set upper-limits on the disk (so-called maximum disks)

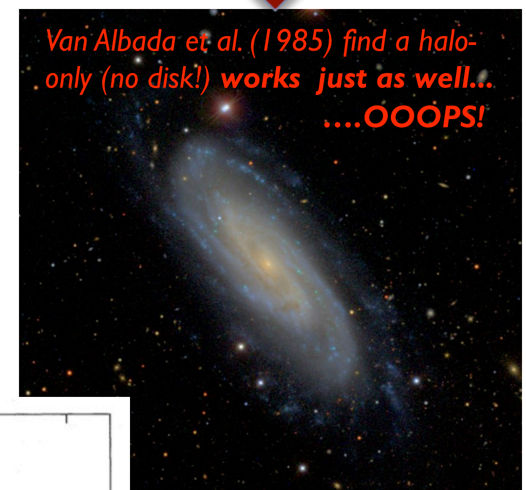
The Disk-Halo Degeneracy: best case

- Rotation curve decomposition constraints:
  - Maximum disk - *yes*   Minimum disk - ***NO***

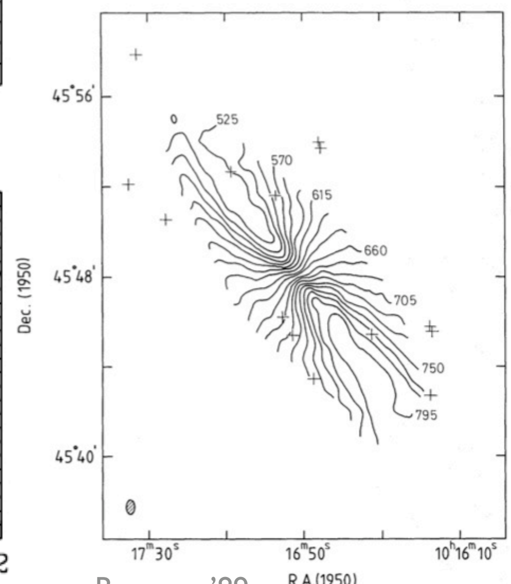


Bershady et al. 2010

← Degenerate solutions…



SDSS gri



Begeman'89

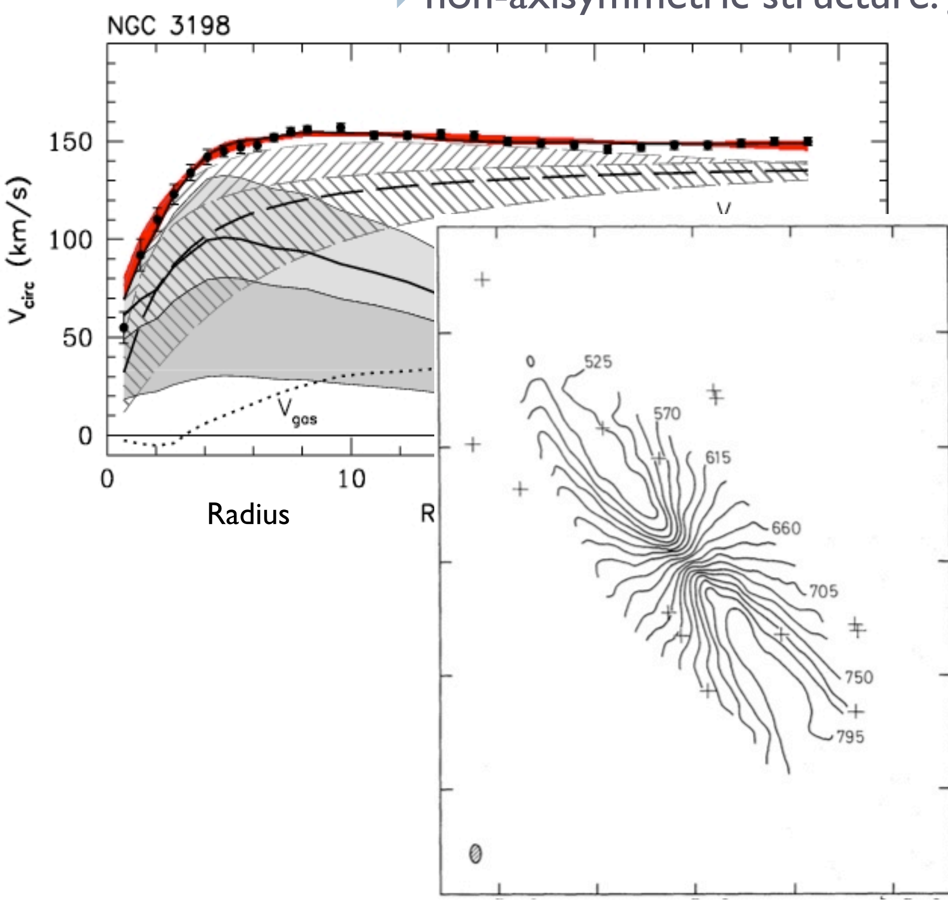
…it doesn't get better than this

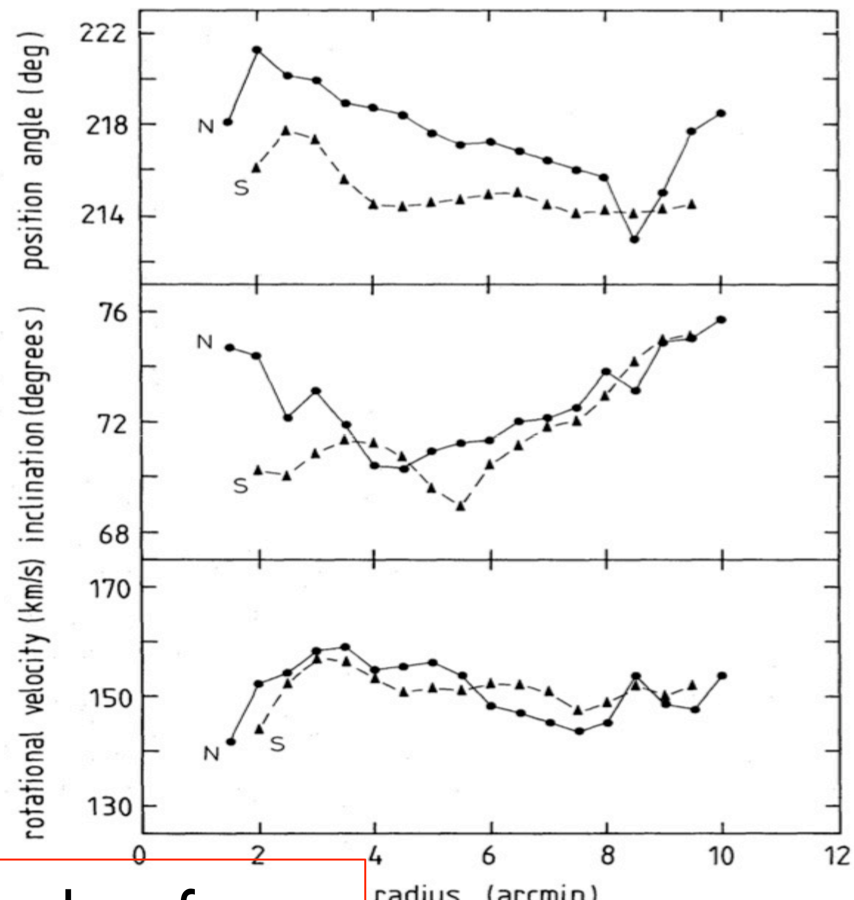
← HI velocity field

Stopped here

The Disk-Halo Degeneracy: Best case

- Formal χ^2 not meaningful at level of ΔV_{circ}<5 km/s
  - non-axisymmetric structure: gas-flows, arms, winds, etc.





Begeman'89

➔ So we need another dynamical probe of mass

Characteristics of dynamical systems - 1

- Summary:
  - $v_c \equiv \sqrt{(r\, d\Phi/dr)} = \sqrt{(GM(r)/r)}$, circular velocity
  - $v_e \equiv (2|\Phi|)^{1/2}$, escape velocity
  - $t_{dyn} \equiv \sqrt{(3\pi/16G\rho)}$
  - $t_{ff} \equiv \sqrt{(1/G\rho)}$, free-fall time $\sim t_{dyn}$
  - $t_{cross} \equiv R/v$, use characteristic radius and velocity

Characteristics of dynamical systems - 2

- Relaxation from N-body encounters of *stars*:
  - $t_s \equiv v^3 / (4\pi\, G^2\, m_*^2\, n)$, …time-scale for strong encounters
    - $\sim 4\times10^{12}$ yr $(v/10 \text{ km s}^{-1})^3 (m_*/M_\odot)^{-2} (n/1\text{pc}^{-3})^{-1}$
    - ➔ unimportant except in very dense star systems
  - However, many weak encounters cumulate such that after a time t_{relax}, the amplitude of the perturbed motion of the star is comparable to its initial motion:
  - $t_{relax} \equiv t_s / 2 \ln \Lambda$
    - $\sim 2\times10^{12}$ yr $(v/10 \text{ km s}^{-1})^3 (m_*/M_\odot)^{-2} (n/1 \text{ pc}^{-3})^{-1} (\ln \Lambda)^{-1}$
    - where $\Lambda = b_{max}/b_{min} \sim R/r_s = N/2$ for isolated system of N stars
      - when $\frac{1}{2} N m_* v^2 \sim G(N m_*)^2/2R$ and $r_s = 2Gm_*/V^2$
    - $t_{relax}/t_{cross} \sim N / 6 \ln N/2$
    - Still very large for realistic N (10^{10} to 10^{11} for galaxies)

Characteristics of dynamical systems - 3

- Instabilities to collapse: the Jean's length
  - $c_s \equiv \sqrt{(k_B T / \mu m_H)}$
    - sound-speed for temperature T and mol. mass μm_H
  - $\lambda_J \equiv cs\sqrt{(\pi/G\rho)} \sim c_s t_{ff}$
  - $M_J \equiv (\pi/6)\ \lambda_J^3 \rho = 20\ M_\odot (T/10K)^{3/2}(100cm^{-3}/n)^{1/2}$
  - What this basically says is that regions smaller than the sound-crossing time have time to re-arrange their density structure in response to gravity, and hence are stable *against* gravitational collapse; larger structures are unstable to collapse.
  - It is relevant for setting the mass-scales for star-formation and galaxy formation.

Dynamics of collisionless systems

- Motivation:
  - Circular rotation is too simple and v_c gives us too little information to constrain Φ and hence ρ (e.g., rotation curves)
  - Without Φ and hence ρ we can't understand how mass has assembled and stars have formed
  - Gas is messy because it requires understanding hydrodynamics, and likely magneto-hydrodynamics.
  - At are disposal are stars, nearly collisionless tracers of Φ!
- How we'll proceed:
  - Start with the Continuity Equation (CE)
  - Use CE to motivate the Collissionless Bolztmann Equation (CBE), like CE but with a force term (remember $\nabla\Phi(\mathbf{x})$!)
  - Develop moments of CBE to relate v and σ and higher-order moments of velocity to Φ and ρ.
- Applications to realistic systems and real problems
  - Velocity ellipsoid
  - Asymmetric drift

Don't be intimidated by moment-integrals of differential equations in cylindrical coordinates: follow the terms, and look for physical intuition.

Disk heating
Disk mass
Disk stability

Continuity Equation

- The mass of fluid in closed volume V, fixed in position and shape, bounded by surface S at time t
  - $M(t) = \int \rho(\mathbf{x},t) d^3\mathbf{x}$
- Mass changes with time as
  - $dM/dt = \int (\partial\rho/\partial t) d^3\mathbf{x} = -\int \rho\, \mathbf{v} \bullet d^2\mathbf{S}$
  - mass flowing out area-element d^2S per unit time is $\rho\, \mathbf{v} \bullet d^2\mathbf{S}$

NB: ∂ = partial derivative

- The above equality allows us to write
  - $\int (\partial\rho/\partial t) d^3\mathbf{x} + \int \rho\, \mathbf{v} \bullet d^2\mathbf{S} = 0$
  - $\int [\, \partial\rho/\partial t + \nabla \bullet (\rho\, \mathbf{v})\,]\, d^3\mathbf{x} = 0$

Divergence theorem

- Since true for any volume
  - $\partial\rho/\partial t + \nabla \bullet (\rho\, \mathbf{v}) = 0$ This is CE

In words: the change in density over time (1st term) is a result of a net divergence in the flow of fluid (2nd term). Stars are a collisionless fluid.

Collisionless Boltzmann Equation

- Generalize concept of spatial density ρ to phase-space density $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$, **where** $f(\mathbf{x},\mathbf{v},t)$ is the distribution function (DF)
  - $f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{x}\, d^3\mathbf{v}$ gives the number of stars at a given time in a small volume $d^3\mathbf{x}$ and velocities in the range $d^3\mathbf{v}$
  - The number-density of stars at location **x** is the integral of $f(\mathbf{x},\mathbf{v},t)$ over velocities:
    - $n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$
  - The mean velocity of stars at location x is then given by
    - $\langle \mathbf{v}(\mathbf{x},t) \rangle = \int \mathbf{v}\, f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v} / \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}$

quantities you can measure

S&G notation

$$\nu(\mathbf{x}) \equiv \int f d^3\mathbf{v}$$

$$\overline{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v}$$

Notation we'll adopt

CBE *continued*

- Goal: Find equation such that given $f(\mathbf{x},\mathbf{v},t_0)$ we can calculate $f(\mathbf{x},\mathbf{v},t)$ at any t, and hence our observable quantities $n(\mathbf{x},t)$, $<v(\mathbf{x},t)>$, etc.
  - Let $\mathbf{w} \equiv (\mathbf{x},\mathbf{v}) = (w_1 \ldots w_6)$
  - $\mathbf{w}' \equiv dw / dt = (\mathbf{x}',\mathbf{v}') = (\mathbf{v}, -\nabla\Phi) = (w_1 \ldots w_3, -\nabla\Phi)$
- CE gives:
  - $df / dt + \Sigma_{i=1,6}\, d(fw_i')/dw_i' = 0$ ....but
    - $dv_i/dx_i = 0$: x_i,v_i independent elements of phase-space
    - $dv_i'/dv_i = 0$: $v' = -\nabla\Phi$, and the gradient in the potential does not depend on velocity.
  - $df / dt + \Sigma_{i=1,6}\, w_i'(df / dw_i) = 0$
- $df / dt + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot df / d\mathbf{v}$ CBE

Initial conditions

Introduce the gravitational potential as something that does work on $f(\mathbf{x},\mathbf{v},t)$

$\rho(\mathbf{x},t) \rightarrow f(\mathbf{x},\mathbf{v},t)$

Vector notation